

CLASSIFICATION OF TRANSITIVE VERTEX ALGEBROIDS

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ABSTRACT. We present a classification of transitive vertex algebroids on a smooth variety X carried out in the spirit of Bressler's classification of Courant algebroids. In particular, we compute the class of the stack of transitive vertex algebroids. We define deformations of sheaves of twisted chiral differential operators introduced in [AChM] and use the classification result to describe and classify such deformations. As a particular case, we obtain a localization of Wakimoto modules at non-critical level on flag manifolds.

1. INTRODUCTION

A vertex algebroid is the algebraic structure induced on a subspace $V_0 \oplus V_1$ of a vertex algebra V . The study of vertex algebroids started with [GMS1] where the sheaves of chiral differential operators (CDO) were defined as the enveloping algebras of exact vertex algebroids.

Algebras of chiral differential operators are sheaves of vertex algebras on smooth varieties resembling the associative algebras of differential operators in some respects. One striking difference from the classical prototype is that for some manifolds X no CDO exists; or if there is one, there may be more than one isomorphism class of such sheaves. Speaking in technical language, sheaves of CDO form a stack, whose groupoid of global sections may be empty or have more than one connected component.

In [GMS1] the classification of chiral differential operators was obtained; in particular, it was established that a global sheaf of CDO exists on X if and only if $ch_2(\Omega_X^1) = 0$ where $ch_2(\Omega_X^1)$ is the second graded piece of Chern character of Ω_X^1 .

This result was re-established by Bressler [Bre] in a rather unexpected fashion. He noticed that the notion of a vertex algebroid is related to a well-known notion in differential geometry, a Courant algebroid: the latter is a quasi-classical limit of the former. He obtained a classification of Courant algebroids extending a fixed Lie algebroid and rediscovered the aforementioned obstruction by connecting the existence of a CDO on X with the existence of certain Courant extensions of the Atiyah algebra of the sheaf of 1-forms.

In both these classification problems the obstruction to global existence is a class in $H^2(X, \Omega^2 \rightarrow \Omega^{3,cl})$. This is due to a rather remarkable property of these algebroids: one can "twist" an algebroid \mathcal{A} on $U \subset X$ by a closed 3-form α . To be more precise, let us denote by $\mathcal{VE}xt_{\mathcal{L}}^{(,)}$ (resp. $\mathcal{CE}xt_{\mathcal{L}}^{(,)}$) the stack of vertex (resp. Courant) algebroid extensions of a given Lie algebroid $\mathcal{L} \xrightarrow{\pi} \mathcal{T}_X$ with an invariant pairing \langle , \rangle on $\ker \pi$ (cf. section 2.4). Then the twisting by 3-form action on $\mathcal{VE}xt_{\mathcal{L}}^{(,)}$ and $\mathcal{CE}xt_{\mathcal{L}}^{(,)}$ extends to an action of a certain stack associated to the complex $\Omega_X^2 \rightarrow \Omega_X^{3,cl}$ (cf. [D]) and it makes each of those a torsor over the latter. By standard abstract nonsense, to every such stack \mathcal{S} there corresponds a class !

$cl(\mathcal{S}) \in H^2(X, \Omega^2 \rightarrow \Omega^{3,cl})$ which vanishes precisely when \mathcal{S} has a global object. For example, the obstruction $ch_2(\Omega_X^1)$ above is exactly the class of the stack of exact vertex algebroids on X .

In this article we classify transitive vertex algebroids. Since exact vertex algebroids classified in [GMS1] are, in fact, a particular kind of transitive vertex algebroids (those whose associated Lie algebroid is the tangent sheaf), our classification generalizes that of [GMS1].

In particular, we compute the class of the stack $\mathcal{VExt}_{\mathcal{L}}^{(\cdot)}$. Bressler computes the corresponding class for Courant extensions of X [Bre] and proves that $cl(\mathcal{CExt}_{\mathcal{L}}^{(\cdot)}) = -\frac{1}{2}p_1(\mathcal{L}, \langle , \rangle)$ where $p_1(\mathcal{L}, \langle , \rangle)$ is the Pontryagin class associated with the pair $(\mathcal{L}, \langle , \rangle)$, a generalization of the familiar first Pontryagin class of a vector bundle, defined in *loc.cit..*

Our main result is Theorem 1.1 below. To prove it we take up the techniques of Baer arithmetic developed by Bressler for Courant algebroids and use the classification of both CDO and Courant algebroids.

Theorem 1.1. *The class of $\mathcal{VExt}_{\mathcal{L}}^{(\cdot)}$ in $H^2(X, \Omega^2 \rightarrow \Omega^{3,cl})$ equals*

$$cl(\mathcal{VExt}_{\mathcal{L}}^{(\cdot)}) = ch_2(\Omega_X^1) - \frac{1}{2}p_1(\mathcal{L}, \langle , \rangle)$$

It is worthwhile to note that it is possible for a manifold X to have no global CDO and Courant extensions of a given Lie algebroid \mathcal{L} , but still have a vertex extension of \mathcal{L} .

We use the classification result above to study certain deformations of sheaves of *twisted chiral differential operators* (TCDO) defined in [AChM]. A TCDO is defined through a procedure that, starting with a CDO produces a sheaf which has features of both the original CDO and the Bernstein-Beilinson algebra of twisted differential operators ([BB1]). These sheaves have proved useful in representation theory of affine Lie algebras at the critical level. In particular, one has a localization procedure for certain classes of $\hat{\mathfrak{g}}$ -modules. [AChM].

More explicitly, a sheaf of TCDO on X is a sheaf of vertex algebras that locally looks like $\mathcal{D}^{ch} \otimes H_X$ where \mathcal{D}^{ch} is a sheaf of CDO on X and H_X is the algebra of differential polynomials on the space $H^1(X, \Omega^{1,cl})$ classifying the twisted differential operators on X .

When X is a flag variety, $X = G/B_-$, the algebra $H_{G/B}$ is isomorphic to $\mathbb{C}[\mathfrak{h}^*]$ where \mathfrak{h} is the Cartan subalgebra of $\mathfrak{g} = \text{Lie } G$. Moreover, there is an embedding of affine vertex algebra

$$V_{-h^\vee}(\mathfrak{g}) \rightarrow \Gamma(G/B, \mathcal{D}_{G/B}^{ch, tw})$$

which makes the space of sections of the TCDO over big cell $\Gamma(U_e, \mathcal{D}_X^{ch, tw}) \simeq \mathcal{D}^{ch}(U_e) \otimes H_{G/B}$ a \mathfrak{g} -module of the critical level, called the *Wakimoto module* $W_{0, -h^\vee}$. [FF1]

The Wakimoto module $W_{0, -h^\vee}$ is a member of the family

$$W_{0,k} = \mathcal{D}^{ch}(U_e) \otimes H_{X, k+h^\vee}$$

where $H_{X, \kappa}$ is the Heisenberg vertex algebra associated with the space \mathfrak{h} with a bilinear form equal to κ times the normalized Killing form.

One might ask whether $W_{0,k}$ with non-critical k admits a localization similar to that of $W_{0, -h^\vee}$. We show that such a sheaf indeed exists on any flag manifold and

is, in fact, a deformation of the TCDO mentioned above: there is one such sheaf for each choice of an invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . If $X = \mathbb{P}^1$, then we prove that this sheaf is a sheaf of $\widehat{\mathfrak{sl}}_2$ -modules of level $\langle \cdot, \cdot \rangle + \langle \cdot, \cdot \rangle_{crit}$ (cf. Corollary 4.8).

More generally, we define a *deformation of TCDO* on an arbitrary manifold X to be the vertex enveloping algebra of certain transitive vertex algebroid on X . We apply our main classification result (cf. Theorem 1.1 above) to classify the deformations.

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2. PRELIMINARIES

We will recall the basic notions of vertex algebra following the exposition of [AChM].

All vector spaces will be over \mathbb{C} .

2.1. Definitions and examples. Let V be a vector space.

A *field* on V is a formal series

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in (\text{End } V)[[z, z^{-1}]]$$

such that for any $v \in V$ one has $a_{(n)}v = 0$ for sufficiently large n .

Let $Fields(V)$ denote the space of all fields on V .

A *vertex algebra* is a vector space V with the following data:

- a linear map $Y : V \rightarrow Fields(V)$, $V \ni a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$
- a vector $|0\rangle \in V$, called *vacuum vector*
- a linear operator $\partial : V \rightarrow V$, called *translation operator*

that satisfy the following axioms:

(1) (Translation Covariance)

$$(\partial a)(z) = \partial_z a(z)$$

(2) (Vacuum)

$$|0\rangle(z) = \text{id};$$

$$a(z)|0\rangle \in V[z] \text{ and } a_{(-1)}|0\rangle = a$$

(3) (Borcherds identity)

$$\begin{aligned} (2.1) \quad & \sum_{j \geq 0} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)} \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} \{a_{(m+n-j)} b_{(k+j)} - (-1)^n b_{(n+k-j)} a_{(m+j)}\} \end{aligned}$$

A vertex algebra V is *graded* if $V = \bigoplus_{n \geq 0} V_n$ and for $a \in V_i$, $b \in V_j$ we have

$$a_{(k)} b \in V_{i+j-k-1}$$

for all $k \in \mathbb{Z}$. (We put $V_i = 0$ for $i < 0$.)

All vertex algebras in this article will be graded.

We say that a vector $v \in V_m$ has *conformal weight* m and write $\Delta_v = m$.

If $v \in V_m$ we denote $v_k = v_{(k-m+1)}$, this is the so-called conformal weight notation for operators. One has

$$v_k V_m \subset V_{m-k}.$$

A *morphism* of vertex algebras is a map $f : V \rightarrow W$ that preserves vacuum and satisfies $f(v_{(n)}v') = f(v)_{(n)}f(v')$.

A *module* over a vertex algebra V is a vector space M together with a map

$$(2.2) \quad Y^M : V \rightarrow \text{Fields}(M), \quad a \mapsto Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1},$$

that satisfy the following axioms:

- (1) $|0\rangle^M(z) = \text{id}_M$
- (2) (Borcherds identity)

$$(2.3) \quad \begin{aligned} & \sum_{j \geq 0} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)}^M \\ &= \sum_{j \geq 0} (-1)^j \binom{n}{j} \{ a_{(m+n-j)}^M b_{(k+j)}^M - (-1)^n b_{(n+k-j)}^M a_{(m+j)}^M \} \end{aligned}$$

A module M over a graded vertex algebra V is called *graded* if $M = \bigoplus_{n \geq 0} M_n$ with $v_k M_l \subset M_{l-k}$ (assuming $M_n = 0$ for negative n).

A *morphism of modules* over a vertex algebra V is a map $f : M \rightarrow N$ that satisfies $f(v_{(n)}^M m) = v_{(n)}^N f(m)$ for $v \in V$, $m \in M$. f is *homogeneous* if $f(M_k) \subset N_k$ for all k .

2.1.1. Commutative vertex algebras. A vertex algebra is said to be *commutative* if $a_{(n)} b = 0$ for a, b in V and $n \geq 0$. The structure of a commutative vertex algebras is equivalent to one of commutative associative algebra with a derivation.

If W is a vector space we denote by H_W the algebra of differential polynomials on W . As an associative algebra it is a polynomial algebra in variables x_i , ∂x_i , $\partial^{(2)}x_i$, ... where $\{x_i\}$ is a basis of W^* . A commutative vertex algebra structure on H_W is uniquely determined by attaching the field $x(z) = e^{z\partial}x_i$ to $x \in W^*$.

H_W is equipped with grading such that

$$(2.4) \quad (H_W)_0 = \mathbb{C}, \quad (H_W)_1 = W^*.$$

2.1.2. Beta-gamma system. Define the Heisenberg Lie algebra to be the algebra with generators a_n^i , b_n^i , $1 \leq i \leq N$ and K that satisfy $[a_m^i, b_n^j] = \delta_{m,-n}\delta_{i,j}K$, $[a_n^i, a_m^j] = 0$, $[b_n^i, b_m^j] = 0$.

Its Fock representation M is defined to be the module induced from the one-dimensional representation \mathbb{C}_1 of its subalgebra spanned by a_n^i , $n \geq 0$, b_m^i , $m > 0$ and K with K acting as identity and all the other generators acting as zero.

The *beta-gamma system* has M as an underlying vector space, the vertex algebra structure being determined by assigning the fields

$$a^i(z) = \sum a_n^i z^{-n-1}, \quad b^i(z) = \sum b_n^i z^{-n}$$

to $a_{-1}^i 1$ and $b_0^i 1$ resp., where $1 \in \mathbb{C}_1$.

This vertex algebra is given a grading so that the degree of operators a_n^i and b_n^i is n . In particular,

$$(2.5) \quad M_0 = \mathbb{C}[b_0^1, \dots, b_0^N], \quad M_1 = \bigoplus_{j=1}^N (b_{-1}^j M_0 \oplus a_{-1}^j M_0).$$

2.2. Vertex algebroids.

2.2.1. *Definition.* Let V be a vertex algebra.

Define a 1-truncated vertex algebra to be a sextuple $(V_0 \oplus V_1, |0\rangle, \partial, {}_{(-1)}, {}_{(0)}, {}_{(1)})$ where the operations ${}_{(-1)}, {}_{(0)}, {}_{(1)}$ satisfy all the axioms of a vertex algebra that make sense upon restricting to the subspace $V_0 + V_1$. (The precise definition can be found in [GMS1]). The category of 1-truncated vertex algebras will be denoted $\mathcal{V}\text{ert}_{\leq 1}$.

The definition of vertex algebroid is a reformulation of that of a sheaf of 1-truncated vertex algebras.

Let (X, \mathcal{O}_X) be a space with a sheaf of \mathbb{C} -algebras.

A *vertex \mathcal{O}_X -algebroid* is a sheaf \mathcal{A} of \mathbb{C} -vector spaces equipped with \mathbb{C} -linear maps $\pi : \mathcal{A} \rightarrow \mathcal{T}_X$ and $\partial : \mathcal{O}_X \rightarrow \mathcal{A}$ satisfying $\pi \circ \partial = 0$ and with operations ${}_{(-1)} : \mathcal{O}_X \times \mathcal{A} \rightarrow \mathcal{A}$, ${}_{(0)} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, ${}_{(1)} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{O}_X$ satisfying axioms:

$$\begin{aligned} (2.6) \quad f_{(-1)}(g_{(-1)}v) - (fg)_{(-1)}v &= \pi(v)(f)_{(-1)}\partial(g) + \pi(v)(g)_{(-1)}\partial(f) \\ (2.7) \quad x_{(0)}(f_{(-1)}y) &= \pi(x)(f)_{(-1)}y + f_{(-1)}(x_{(0)}y) \\ (2.8) \quad x_{(0)}y + y_{(0)}x &= \partial(x_{(1)}y) \\ (2.9) \quad \pi(f_{(-1)}v) &= f\pi(v) \\ (2.10) \quad (f_{(-1)}x)_{(1)}y &= f(x_{(1)}y) - \pi(x)(\pi(y)(f)) \\ (2.11) \quad \pi(v)(x_{(1)}y) &= (v_{(0)}x)_{(1)}y + x_{(1)}(v_{(0)}y) \\ (2.12) \quad \partial(fg) &= f_{(-1)}\partial(g) + g_{(-1)}\partial(f) \\ (2.13) \quad v_{(0)}\partial(f) &= \partial(\pi(v)(f)) \\ (2.14) \quad v_{(1)}\partial(f) &= \pi(v)(f) \end{aligned}$$

for $v, x, y \in \mathcal{A}$, $f, g \in \mathcal{O}_X$. The map π is called the *anchor* of \mathcal{A} .

If $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n$ is a (graded) sheaf of vertex algebras with $\mathcal{V}_0 = \mathcal{O}_X$, then $\mathcal{A} = \mathcal{V}_1$ is a vertex algebroid with ∂ equal to the translation operator and π sending $x \in \mathcal{V}_1$ to the derivation $f \mapsto x_{(0)}f$.

2.2.2. *Associated Lie algebroid.* Recall that a *Lie algebroid* is a sheaf of \mathcal{O}_X -modules \mathcal{L} equipped with a Lie algebra bracket $[,]$ and a morphism $\pi : \mathcal{A} \rightarrow \mathcal{T}_X$ of Lie algebra and \mathcal{O}_X -modules called *anchor* that satisfies $[x, ay] = a[x, y] + \pi(x)(a)y$, $x, y \in \mathcal{A}$, $a \in \mathcal{O}_X$.

If \mathcal{A} is a vertex algebroid, then the operation ${}_{(0)}$ descends to that on $\mathcal{L}_{\mathcal{A}} = \mathcal{A}/\mathcal{O}_X{}_{(-1)}\partial\mathcal{O}_X$ and makes it into a Lie algebroid, with the anchor induced by that of \mathcal{A} . $\mathcal{L}_{\mathcal{A}}$ is called *the associated Lie algebroid of \mathcal{A}* .

2.2.3. A vertex (resp., Lie) algebroid is *transitive*, if its anchor map π is surjective.

Being a derivation, see (2.12), $\partial : \mathcal{O}_X \rightarrow \mathcal{A}$ lifts to $\Omega_X^1 \rightarrow \mathcal{A}$. It follows from (2.14) that if \mathcal{A} is transitive, then $\Omega_X^1 \simeq \mathcal{O}_X{}_{(-1)}\partial\mathcal{O}_X$ and \mathcal{A} fits into an exact sequence

$$0 \longrightarrow \Omega_X^1 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L} \longrightarrow 0,$$

$\mathcal{L} = \mathcal{L}_{\mathcal{A}}$ being an extension

$$0 \longrightarrow \mathfrak{h}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow \mathcal{T}_X \longrightarrow 0$$

where $\mathfrak{h}(\mathcal{L}) := \ker(\mathcal{L} \xrightarrow{\pi} \mathcal{T}_X)$ is an \mathcal{O}_X -Lie algebra.

Note that the pairing ${}_{(1)}$ on \mathcal{A} induces a symmetric $\mathcal{L}_{\mathcal{A}}$ -invariant \mathcal{O}_X -bilinear pairing on $\mathfrak{g}(\mathcal{L}_{\mathcal{A}})$ which will be denoted by \langle , \rangle .

We regard the pair $(\mathcal{L}_A, \langle , \rangle)$ as "classical data" underlying the vertex algebroid A .

2.2.4. Truncation and vertex enveloping algebra functors. There is an obvious truncation functor

$$t : \mathcal{V}ert \rightarrow \mathcal{V}ert_{\leq 1}$$

that assigns to every vertex algebra a 1-truncated vertex algebra. This functor admits a left adjoint [GMS1]

$$u : \mathcal{V}ert_{\leq 1} \rightarrow \mathcal{V}ert$$

called a *vertex enveloping algebra functor*.

These functors have evident sheaf versions. In particular, one has the functor

$$(2.15) \quad U : \mathcal{V}ertAlg \longrightarrow Sh\mathcal{V}ert$$

from the category of vertex algebroids to the category of sheaves of vertex algebras.

2.3. Courant algebroids. We give a definition of a Courant algebroid following [Bre]; see also [LWX].

A *Leibniz algebra* over k is a k -vector space A with a bracket $[,] : A \otimes_k A \rightarrow A$ satisfying

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

The bracket is not assumed to be skew-commutative.

A *Courant \mathcal{O}_X -algebroid* is an \mathcal{O}_X -module \mathcal{Q} equipped with

- (1) a structure of a Leibniz \mathbb{C} -algebra $[,] : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \rightarrow \mathcal{Q}$,
- (2) an \mathcal{O}_X -linear map of Leibniz algebras (the anchor map) $\pi : \mathcal{Q} \rightarrow \mathcal{T}_X$,
- (3) a symmetric \mathcal{O}_X -bilinear pairing $\langle , \rangle : \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{Q} \rightarrow \mathcal{O}_X$,
- (4) a derivation $\partial : \mathcal{O}_X \rightarrow \mathcal{Q}$

which satisfy

$$(2.16) \quad \pi \circ \partial = 0$$

$$(2.17) \quad [q_1, f q_2] = f[q_1, q_2] + \pi(q_1)(f)q_2$$

$$(2.18) \quad \langle [q, q_1], q_2 \rangle + \langle q_1, [q, q_2] \rangle = \pi(q)(\langle q_1, q_2 \rangle)$$

$$(2.19) \quad [q, \partial(f)] = \partial(\pi(q)(f))$$

$$(2.20) \quad \langle q, \partial(f) \rangle = \pi(q)(f)$$

$$(2.21) \quad [q_1, q_2] + [q_2, q_1] = \partial(\langle q_1, q_2 \rangle)$$

for $f \in \mathcal{O}_X$ and $q, q_1, q_2 \in \mathcal{Q}$.

A morphism of Courant \mathcal{O}_X -algebroids is an \mathcal{O}_X -linear map of Leibnitz algebras which commutes with the respective anchor maps and derivations and preserves the respective pairings.

A *connection* on a Courant algebroid \mathcal{Q} is an \mathcal{O}_X -linear section $\bar{\nabla}$ of the anchor map such that $\langle \bar{\nabla}(\xi), \bar{\nabla}(\eta) \rangle = 0$.

If \mathcal{Q} is a Courant algebroid, then $\mathcal{L}_{\mathcal{Q}} = \mathcal{Q}/\mathcal{O}_X \partial \mathcal{O}_X$ is a Lie algebroid; it is called *the associated Lie algebroid of \mathcal{Q}* . The pairing \langle , \rangle on \mathcal{Q} induces a $\mathcal{L}_{\mathcal{Q}}$ -invariant pairing on $\mathfrak{g}(\mathcal{L}_{\mathcal{Q}})$ which will be denoted \langle , \rangle .

2.4. The category of vertex extensions. Let \mathcal{L} be a transitive Lie algebroid. A *vertex extension* of \mathcal{L} is a vertex algebroid \mathcal{A} with an isomorphism of Lie algebroids $\phi : \mathcal{L}_{\mathcal{A}} \rightarrow \mathcal{L}$. In what follows we will always identify $\mathcal{L}_{\mathcal{A}}$ and \mathcal{L} via ϕ .

A morphism of vertex extensions of \mathcal{L} is a morphism of vertex algebroids $f : \mathcal{A} \rightarrow \mathcal{A}'$ which induces the identity map on \mathcal{L} . Thus f fits into a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{L} & \longrightarrow 0 \\ & & \parallel & & \downarrow f & & \parallel \\ 0 & \longrightarrow & \Omega_X^1 & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{L} & \longrightarrow 0 \end{array}$$

Vertex extensions of \mathcal{L} on X form a category $\mathcal{VE}xt_{\mathcal{L}}(X)$; clearly, it is a groupoid.

One can consider the category of vertex extensions of $\mathcal{L}|_U$ on U for any open subset $U \subset X$. These categories with the obvious restriction functors form a stack on the Zariski topology of X , to be denoted $\mathcal{VE}xt_{\mathcal{L}}$

Let \mathcal{A} be a vertex extension of \mathcal{L} . Denote $\tilde{\mathfrak{g}}_{\mathcal{A}} := \ker(\pi : \mathcal{A} \rightarrow \mathcal{T}_X)$; it is an extension

$$0 \longrightarrow \Omega_X^1 \longrightarrow \tilde{\mathfrak{g}}_{\mathcal{A}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

It is easy to see that the operation ${}_{(1)}$ satisfies $\tilde{\mathfrak{g}}_{\mathcal{A}(1)}\Omega^1 = 0$, and, therefore, induces a (symmetric, \mathcal{O}_X -bilinear) pairing

$$\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathcal{O}_X$$

If $f : \mathcal{A} \rightarrow \mathcal{A}'$ is a morphism of extensions, f induces the identity map on \mathfrak{g} ; it also preserves ${}_{(1)}$. Therefore \mathcal{A} and \mathcal{A}' must have the same pairing \langle , \rangle on \mathfrak{g} . It follows that the groupoid $\mathcal{VE}xt_{\mathcal{L}}(X)$ is a disjoint union

$$\mathcal{VE}xt_{\mathcal{L}}(X) = \coprod_{\langle , \rangle} \mathcal{VE}xt_{\mathcal{L}}^{(\langle , \rangle)}(X)$$

where $\mathcal{VE}xt_{\mathcal{L}}^{(\langle , \rangle)}(X)$ is the full subcategory of vertex extensions of \mathcal{L} whose induced pairing on \mathfrak{g} is \langle , \rangle . Such extensions will be called *vertex extensions of $(\mathcal{L}, \langle , \rangle)$* .

Similarly, we define the notion of a *Courant extension* of \mathcal{L} on X and that of a morphism of Courant extensions, the categories $\mathcal{CE}xt_{\mathcal{L}}^{(\langle , \rangle)}(U)$ and $\mathcal{CE}xt_{\mathcal{L}}^{(\langle , \rangle)}(U)$, $U \subset X$.

2.5. Chiral differential operators. Vertex extensions of \mathcal{T}_X are called *exact vertex algebroids*. Their vertex enveloping algebras, sheaves of *chiral differential operators* (CDO), were first introduced in [MSV] and classified in [GMS1]. Let us recall the main classification result.

Let us call a smooth affine variety $U = \text{Spec } A$ *suitable for chiralization* if $\text{Der}(A)$ is a free A -module admitting an abelian frame $\{\tau_1, \dots, \tau_n\}$. In this case there is a CDO over U , which is uniquely determined by the condition that $(\tau_i)_{(1)}(\tau_j) = (\tau_i)_{(0)}(\tau_j) = 0$. Denote this CDO by $D_{U, \tau}^{ch}$.

Theorem 2.1. *Let $U = \text{Spec } A$ be suitable for chiralization with a fixed abelian frame $\{\tau_i\} \subset \text{Der } A$.*

(i) *For each closed 3-form $\alpha \in \Omega_A^{3,cl}$ there is a CDO over U that is uniquely determined by the conditions*

$$(\tau_i)_{(1)}\tau_j = 0, \quad (\tau_i)_{(0)}\tau_j = \iota_{\tau_i}\iota_{\tau_j}\alpha.$$

Denote this CDO by $\mathcal{D}_{U, \tau}(\alpha)$.

(ii) *Each CDO over U is isomorphic to $\mathcal{D}_{U, \tau}(\alpha)$ for some α .*

(iii) $\mathcal{D}_{U,\tau}(\alpha_1)$ and $\mathcal{D}_{U,\tau}(\alpha_2)$ are isomorphic if and only if there is $\beta \in \Omega_A^2$ such that $d\beta = \alpha_1 - \alpha_2$. In this case the isomorphism is determined by the assignment $\tau_i \mapsto \tau_i + \iota_{\tau_i}\beta$.

If $A = \mathbb{C}[x_1, \dots, x_n]$, one can choose $\partial/\partial x_j$, $j = 1, \dots, n$, for an abelian frame and check that the beta-gamma system M of sect. 2.1.2 is a unique up to isomorphism CDO over \mathbb{C}^n . A passage from M to Theorem 2.1 is accomplished by the identifications $b_0^j 1 = x_j$, $a_{-1}^j 1 = \partial/\partial x_j$.

3. CLASSIFICATION OF TRANSITIVE VERTEX ALGEBROIDS

In this section we present a classification of transitive vertex algebroids in the spirit of [Bre].

First we recall the definition of a $\mathcal{G}r(\Omega_X^{[2,3]})$ -gerbe given in [GMS1]; one of the results of [Bre] is that $\mathcal{V}\mathcal{E}xt_{\mathcal{L}}^{(,)}$ is a $\mathcal{G}r(\Omega_X^{[2,3]})$ -gerbe.

In section 3.3 we describe the core tool of the classification method: the “addition” operation on various algebroids. It enables us to construct a vertex extension starting from a Courant extension and an exact vertex algebroid. With this tool in hands we are able to compute the class of the stack $\mathcal{V}\mathcal{E}xt_{\mathcal{L}}^{(,)}$ (Theorem 3.14).

3.1. Gerbes and torsors.

3.1.1. *Twisting by a 3-form.* Let $\mathcal{A} = (\mathcal{A}, (-1), (0), (1), \partial, \pi)$ be a vertex extension of \mathcal{L} on $U \subset X$ and let $\alpha \in \Omega^{3,cl}(U)$. Define an operation $(0)+\alpha : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ by

$$(3.1) \quad x(0)+\alpha y = x(0)y + \iota_{\pi(x)}\iota_{\pi(y)}\alpha$$

Lemma 3.1. *Let $\alpha \in \Omega^{3,cl}(U)$. Then:*

- (1) $\mathcal{A} + \alpha := (\mathcal{A}, (-1), (0)+\alpha, (1), \partial, \pi)$ is a vertex extension of \mathcal{L} on U .
- (2) The assignment $\mathcal{A} \mapsto \mathcal{A} + \alpha$ can be extended to an auto-equivalence

$$(3.2) \quad ? + \alpha : \mathcal{V}\mathcal{E}xt_{\mathcal{L}}^{(,)} \rightarrow \mathcal{V}\mathcal{E}xt_{\mathcal{L}}^{(,)}$$

Proof. The proof of (1) is the same as in the case of cdo ([MSV, GMS1]) or Courant algebroids ([Bre]). To see (2), note that every morphism $f : \mathcal{A} \rightarrow \mathcal{A}'$ is automatically a morphism $\mathcal{A} + \alpha \rightarrow \mathcal{A}' + \alpha$; this tautological action on morphisms makes $? + \alpha$ a functor; the composition $(? + (-\alpha)) \circ (? + \alpha)$ is the identity functor of $\mathcal{V}\mathcal{E}xt_{\mathcal{L}}^{(,)}$. \square

It is clear that the functors $? + \alpha$, $\alpha \in \Omega^{3,cl}(U)$ define an action of the abelian group $\Omega^{3,cl}(U)$ on the category $\mathcal{V}\mathcal{E}xt_{\mathcal{L}}(U)$. Let us show that this action in fact extends to an action of a category.

For an open subset $U \subset X$ define a category $\mathcal{G}r(\Omega^{[2,3]})(U)$ as follows. The objects of $\mathcal{G}r(\Omega^{[2,3]})(U)$ are elements $a \in \Omega^{3,cl}(U)$; the morphisms

$$\text{Hom}(\alpha, \alpha') = \{\beta \in \Omega^2(U) : d\beta = \alpha' - \alpha\},$$

the composition being the addition in $\Omega^2(U)$.

It is clear that $\mathcal{G}r(\Omega^{[2,3]})(U)$ is a groupoid. The groupoids $\mathcal{G}r(\Omega^{[2,3]})(U)$ form a prestack $\mathcal{G}r(\Omega_X^{[2,3]})$; the addition of 3-forms gives it the structure of a *Picard prestack*. See [D], section 1.4 for generalities on Picard stacks.

For $f : \mathcal{A} \rightarrow \mathcal{A}'$ and $\beta : \alpha \rightarrow \alpha'$ define

$$(3.3) \quad (f + \beta)(x) = f(x) + \iota_{\pi(x)}\beta$$

Proposition 3.2. (1) $f \dotplus \beta$ is a morphism of vertex extensions

$$f \dotplus \beta : \mathcal{A} \dotplus \alpha \rightarrow \mathcal{A}' \dotplus \alpha'$$

(2) The formulas (3.2) and (3.3) define a functor

$$+ : \mathcal{VE}xt_{\mathcal{L}}(U) \times \mathcal{Gr}(\Omega^{[2,3]})_{\text{loc}}(U) \longrightarrow \mathcal{VE}xt_{\mathcal{L}}(U)$$

which gives rise to an action of $\mathcal{Gr}(\Omega^{[2,3]})_{\text{loc}}(U)$ on $\mathcal{VE}xt_{\mathcal{L}}(U)$

The verification is, again, straightforward and repeats the analogous discussion in [GMS1]. \square

3.1.2. $(\Omega^2 \rightarrow \Omega^{3,cl})$ -gerbes. We will say a stack \mathcal{S} over X is a $\mathcal{Gr}(\Omega_X^{[2,3]})$ -gerbe if there is an action $\dotplus : \mathcal{S} \times \mathcal{Gr}(\Omega_X^{[2,3]}) \rightarrow \mathcal{S}$ and a cover $\mathfrak{U} = \{U_i\}_{i \in \mathcal{I}}$ such that for any $i \in \mathcal{I}$ and $x \in \mathcal{S}(U_i)$ the functor $x \dotplus ? : \mathcal{Gr}(\Omega_X^{[2,3]})_{\text{loc}}(U_i) \rightarrow \mathcal{S}(U_i)$ is an equivalence. (In other words, \mathcal{S} is a torsor over the associated stack).

Theorem 3.3. [Bre] The stacks $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$ and $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}$, when locally nonempty, are $\mathcal{Gr}(\Omega^{[2,3]})$ -gerbes.

Remark 3.4. The categories $\mathcal{Gr}(\Omega^{[2,3]})_{\text{loc}}(U)$, $U \subset X$ form a Picard prestack (cf.[D], section 1.4.11) whose associated stack is the stack of $(\Omega^2 \rightarrow \Omega^{3,cl})$ -torsors.

What Bressler shows in [Bre] is that this stack is equivalent to the stack \mathcal{ECA}_X of exact Courant algebroids, and that the stacks $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$, $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}$ are, in fact, \mathcal{ECA}_X -torsors.

Observe that for $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$ being a $\mathcal{Gr}(\Omega^{[2,3]})$ -gerbe means that for small enough $U \subset X$ and $\mathcal{A} \in \mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$ one has an equivalence

$$\mathcal{A} \dotplus ? : \mathcal{Gr}(\Omega^{[2,3]})_{\text{loc}}(U) \rightarrow \mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}(U)$$

In particular, there is an isomorphism

$$\text{Hom}(\alpha, \alpha') \simeq \text{Hom}(\mathcal{A} + \alpha, \mathcal{A} + \alpha')$$

Under this isomorphism, an element $\beta \in \Omega^2$ with $d\beta = \alpha' - \alpha$, is mapped to the morphism (cf. (3.3))

$$(3.4) \quad \exp(\beta) := \text{id} \dotplus \beta : x \mapsto x + \iota_{\pi(x)}\beta$$

The same is true for Courant algebroids and we will use the notation $\exp(\beta)$ in both cases.

3.1.3. The class of a gerbe. Let \mathcal{S} be a $\mathcal{Gr}(\Omega^{[2,3]})$ -gerbe and \mathfrak{U} a cover as in 3.1.2. Let us choose an object $x_i \in \mathcal{S}(U_i)$ for each i . For each pair i, j we have objects $x_i|_{U_{ij}}$ and $x_j|_{U_{ij}}$, and therefore, an isomorphism

$$(3.5) \quad \eta_{ij} : x_i|_{U_{ij}} \rightarrow x_j|_{U_{ij}} \dotplus \alpha_{ij}$$

for some $\alpha_{ij} \in \Omega^{3,cl}$.

The collection $(x_i, \eta_{ij}, \alpha_{ij})$ is called a trivialization of \mathcal{S} .

We will denote by the same letter η_{ij} all of its translates

$$\eta_{ij} \dotplus \text{id}_{\gamma} : x_i|_{U_{ij}} \dotplus \gamma \rightarrow x_j|_{U_{ij}} \dotplus (\alpha_{ij} + \gamma)$$

for $\gamma \in \Omega^{3,cl}(U_i)$.

For each triple i, j, k consider the composition (over $U_{ijk} = U_i \cap U_j \cap U_k$)

$$\eta_{jk}\eta_{ij}\eta_{ik}^{-1} : x_k \xrightarrow{\eta_{ik}^{-1}} x_i + (-\alpha_{ik}) \xrightarrow{\eta_{ij}} x_j + (\alpha_{ij} - \alpha_{ik}) \xrightarrow{\eta_{jk}} x_k + (\alpha_{ij} + \alpha_{jk} - \alpha_{ik})$$

and denote by β_{ijk} the element of $\Omega^2(U_{ijk})$ such that

$$(3.6) \quad \eta_{jk}\eta_{ij}\eta_{ik}^{-1} = \exp(\beta_{ijk})$$

One checks that

$$(3.7) \quad d_{\check{C}}\beta_{ijk} = 0, \quad d_{DR}(\beta_{ijk}) = d_{\check{C}}(\alpha_{ij}), \quad d_{DR}(\alpha_{ij}) = 0$$

so that the pair $(\alpha_{ij}, \beta_{ijk})$ is an element of $\check{Z}^2(\mathfrak{U}, \Omega^2 \rightarrow \Omega^{3,cl})$.

By definition, the *class* of \mathcal{S} , $cl(\mathcal{S})$, is the class of $(\alpha_{ij}, \beta_{ijk})$ in $H^2(X, \Omega^2 \rightarrow \Omega^{3,cl})$. One has the following classical result (cf., e.g., [GMS1] for a proof).

Proposition 3.5. $\mathcal{S}(X)$ is nonempty if and only if $cl(\mathcal{S}) = 0$. \square

3.2. The stack $\mathcal{CE}xt_{\mathcal{L}}^{<: >}$. As an example, and for future use, we recall the construction of a trivialization of the stack $\mathcal{CE}xt_{\mathcal{L}}^{<: >}$ given in [Bre].

Let us choose a cover $\mathfrak{U} = \{U_i\}$ such that \mathcal{T}_{U_i} is free, choose connections (\mathcal{O}_X -linear sections of the anchor map)

$$\nabla_i : \mathcal{T}_{U_i} \rightarrow \mathcal{L}_{U_i}$$

and identify $\mathcal{L}_{U_i} \simeq \mathcal{T}_{U_i} \oplus \mathfrak{g}_{U_i}$ via ∇_i .

Define $c_i = c(\nabla_i) \in \Omega_{U_i}^{2,cl} \otimes_{\mathcal{O}} \mathfrak{g}_{U_i}$ to be the *curvature* of the connection ∇_i , i.e.

$$c_i(\xi, \eta) = [\nabla_i(\xi), \nabla_i(\eta)] - \nabla_i([\xi, \eta])$$

Recall the following

Theorem 3.6. [Bre] Let $U \subset X$ and $\nabla : \mathcal{T}_U \rightarrow \mathcal{L}_U$ is any connection.

Then the category $\mathcal{CE}xt_{\mathcal{L}}^{<: >}(U)$ is nonempty if and only if the form $\frac{1}{2}\langle c(\nabla) \wedge c(\nabla) \rangle$ is exact.

Assume that

$$\frac{1}{2}\langle c(\nabla_i) \wedge c(\nabla_i) \rangle = dH_i$$

for some $H_i \in \Omega^3$. Then one can construct a Courant extension $\mathcal{Q}_{\nabla_i, H_i}$, which is equal to $\mathcal{L}_{U_i} \oplus \Omega_{U_i}^1$ as a sheaf of \mathcal{O}_U -modules, and satisfies

$$(3.8) \quad [\xi, \eta] = [\xi, \eta]_{\mathcal{L}} + \iota_{\xi}\iota_{\eta}H_i, \quad \xi, \eta \in \mathcal{T}_{U_i},$$

$$(3.9) \quad \langle \mathfrak{g}, \Omega_U^1 \rangle = \langle \mathfrak{g}, \nabla_i(\mathcal{T}_U) \rangle = 0,$$

$$(3.10) \quad [\xi, g] = [\nabla_i(\xi), g]_{\mathcal{L}} - \langle \iota_{\xi}c(\nabla_i), g \rangle.$$

For each i, j define

$$A_{ij} = \nabla_i - \nabla_j \in \Omega_{U_{ij}}^1 \otimes \mathfrak{g}_{U_{ij}}$$

Theorem 3.7. [Bre] *There exists an isomorphism in $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U_{ij})$*

$$(3.11) \quad \theta_{ij} : \mathcal{Q}_{\nabla_i, H_i} \xrightarrow{\sim} \mathcal{Q}_{\nabla_j, H_j} + \alpha_{ij}$$

given by

$$(3.12) \quad \begin{aligned} \xi &\mapsto \xi + A_{ij}(\xi) - \frac{1}{2}\langle A_{ij}(\xi), A_{ij} \rangle \\ g &\mapsto g - \langle g, A_{ij} \rangle \\ \omega &\mapsto \omega \end{aligned}$$

where

$$(3.13) \quad \alpha_{ij} = \langle c(\nabla_i) \wedge A_{ij} \rangle - \frac{1}{2}\langle [\nabla_i, A_{ij}], A_{ij} \rangle + \frac{1}{6}\langle [A_{ij}, A_{ij}], A_{ij} \rangle + H_i - H_j$$

The collection $(\mathcal{Q}_{\nabla_i, H_i}, \theta_{ij}, \alpha_{ij})$ is a trivialization of the gerbe $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}$.

On triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ the isomorphisms θ_{ij} satisfy ([Bre])

$$(3.14) \quad \theta_{jk}\theta_{ij}\theta_{ik}^{-1} = \exp(-\langle A_{ij} \wedge A_{jk} \rangle) \quad \square$$

Define $\beta_{ijk} = -\langle A_{ij} \wedge A_{jk} \rangle$.

Then $(\alpha_{ij}, \beta_{ijk})$ is a cocycle in $\check{Z}^2(\mathfrak{U}, \Omega_X^2 \rightarrow \Omega_X^{3,cl})$. The corresponding cohomology class was identified in [Bre] with minus one half of the first Pontryagin class $p_1(\mathcal{L}, \langle , \rangle)$ of $(\mathcal{L}, \langle , \rangle)$.

Theorem 3.8. [Bre] *This class is the class of the stack $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}$:*

$$cl(\mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}) = -\frac{1}{2}p_1(\mathcal{L}, \langle , \rangle).$$

3.3. Linear algebra. In this section we describe the main tool in the proof of the classification result: we define linear algebra-like operations on various algebroids. The main technical result to be proved in this section is as follows.

Theorem 3.9. *Let U be suitable for chiralization.*

Then there exist a functor

$$\boxplus : \mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U) \times \mathcal{CD}\mathcal{O}(U) \longrightarrow \mathcal{VE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U)$$

$$(\mathcal{Q}, \mathcal{D}) \mapsto \mathcal{Q} \boxplus \mathcal{D}$$

and a functor

$$\boxminus : \mathcal{VE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U) \times \mathcal{CD}\mathcal{O}(U) \longrightarrow \mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U)$$

$$(\mathcal{A}, \mathcal{D}) \mapsto \mathcal{A} \boxminus \mathcal{D}$$

such that for a fixed $\mathcal{D} \in \mathcal{CD}\mathcal{O}(U)$ the functors

$$-\boxminus \mathcal{D} : \mathcal{VE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U) \rightarrow \mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U)$$

and

$$-\boxplus \mathcal{D} : \mathcal{CE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U) \rightarrow \mathcal{VE}xt_{\mathcal{L}}^{(\cdot,\cdot)}(U)$$

are mutually inverse equivalences of $Gr(\Omega^{[2,3]}(U))$ -torsors.

In fact, the functor \boxminus was defined in [Bre], together with several versions of \boxplus defined for various algebroids. Our \boxplus is just an extension of Bressler's definition.

3.3.1. *Addition.* Let \mathcal{Q} be a Courant extension of \mathcal{L} and \mathcal{D} a cdo.

We describe how to define a vertex extension of \mathcal{L} which can be thought of as a “sum” of these two structures; the construction parallels that of the Baer sum of two extensions.

First, consider the pullback $\overline{\mathcal{A}} := \mathcal{Q} \times_{\mathcal{T}} \mathcal{D}$ so that a section of \mathcal{A} is a pair (q, x) , $q \in \mathcal{Q}$, $x \in \mathcal{D}$ with $\pi(q) = \pi(x)$.

Define operations ${}_{(-1)} : \mathcal{O}_X \times \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}}$ and ${}_{(0)}, {}_{(1)} : \overline{\mathcal{A}} \times \overline{\mathcal{A}} \rightarrow \overline{\mathcal{A}}$ as follows:

$$(3.15) \quad a_{(-1)}(q, x) := (aq, a_{(-1)}x)$$

$$(3.16) \quad (q, x)_{(0)}(q', x') := ([q, q']_{\mathcal{Q}}, x_{(0)}x'),$$

$$(3.17) \quad (q, x)_{(1)}(q', x') := \langle q, q' \rangle + x_{(1)}x',$$

$$(3.18) \quad \pi((q, x)) := \pi(q) = \pi(x),$$

$$(3.19) \quad \partial a = (\partial a, 0)$$

Note that $\overline{\mathcal{A}}$ contains two copies of Ω^1 , one from \mathcal{Q} and the other from \mathcal{D} .

Let us define $\mathcal{Q} \boxplus \mathcal{D}$ to be the pushout of $\overline{\mathcal{A}}$ with respect to the addition map $+ : \Omega^1 \times \Omega^1 \rightarrow \Omega^1$ so that one has the following

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^1 \oplus \Omega^1 & \longrightarrow & \overline{\mathcal{A}} & \longrightarrow & \mathcal{L} \longrightarrow 0 \\ & & \downarrow + & & \downarrow & & \parallel \\ 0 & \longrightarrow & \Omega^1 & \longrightarrow & \mathcal{Q} \boxplus \mathcal{D} & \longrightarrow & \mathcal{L} \longrightarrow 0 \end{array}$$

Alternatively, $\mathcal{Q} \boxplus \mathcal{D}$ fits into the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathfrak{g}} \oplus \Omega^1 & \longrightarrow & \overline{\mathcal{A}} & \xrightarrow{\pi} & \mathcal{T}_X \longrightarrow 0 \\ & & \downarrow + & & \downarrow & & \parallel \\ 0 & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathcal{Q} \boxplus \mathcal{D} & \longrightarrow & \mathcal{T}_X \longrightarrow 0 \end{array}$$

where the rows are exact and the left square is a push-out square.

Theorem 3.10. *The operations (3.15 - 3.19) make sense on $\mathcal{Q} \boxplus \mathcal{D}$ and give it the structure of a vertex algebroid*

Proof. The verification is straightforward. As an example, let us show that (2.10) is satisfied.

For $f \in \mathcal{O}_X$, $q \in \mathcal{Q}$, $v \in \mathcal{D}$, one has:

$$\begin{aligned} (f_{(-1)}(q, v))_{(1)}(q', v') &= (fq, f_{(-1)}v)_{(1)}(q', v') = \langle fq, q' \rangle + (f_{(-1)}v)_{(1)}v' \\ &= f\langle q, q' \rangle + f(v_{(1)}v') - \pi(v)\pi(v')(f) = f((q, v)_{(1)}(q', v')) - \pi((q, v))\pi((q', v'))(f) \end{aligned}$$

□

Note that the assignment $(\mathcal{Q}, \mathcal{D}) \mapsto \mathcal{Q} \boxplus \mathcal{D}$ is naturally a functor

$$\boxplus : \mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}(U) \times \mathcal{CD}\mathcal{O}(U) \longrightarrow \mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}(U)$$

Indeed, let $f \in \text{Hom}_{\mathcal{CE}xt}(\mathcal{Q}, \mathcal{Q}')$, $g \in \text{Hom}_{\mathcal{CD}\mathcal{O}}(\mathcal{D}, \mathcal{D}')$. In particular, f and g are maps over \mathcal{T} , so (f, g) takes $\mathcal{Q} \times_{\mathcal{T}} \mathcal{D} \subset \mathcal{Q} \times \mathcal{D}$ to $\mathcal{Q}' \times_{\mathcal{T}} \mathcal{D}'$. Since f and g act as identity on the subsheaf Ω^1 , (f, g) gives a well-defined map between the pushouts $\mathcal{Q} \boxplus \mathcal{D} \rightarrow \mathcal{Q}' \boxplus \mathcal{D}'$ that will be denoted $f \boxplus g$. Finally, it remains to note that the composition is “coordinate-wise”:

$$(3.20) \quad (f \boxplus g)(f' \boxplus g') = ff' \boxplus gg'$$

which implies that $(f, g) \mapsto f \boxplus g$ is a functor.

Let us note, among the elementary properties of this functor, the following:

(1) for any $\alpha \in \Omega_U^{3,cl}$, $\mathcal{Q} \in \mathcal{CExt}_{\mathcal{L}}^{(\cdot)}(U)$, $\mathcal{D} \in \mathcal{CD}\mathcal{O}_U$ one has the equalities

$$(3.21) \quad (\mathcal{Q} \boxplus \mathcal{D}) + \alpha \cong \mathcal{Q} \boxplus (\mathcal{D} + \alpha) \cong (\mathcal{Q} + \alpha) \boxplus \mathcal{D}$$

(by definition of $+ \alpha$ the three parts of the equation have underlying sheaf $\mathcal{Q} \boxplus \mathcal{D}$, one only has to check that the operations coincide).

(2) one has the equality

$$\exp(\beta) \boxplus \text{id}_{\mathcal{D}} = \exp(\beta) = \text{id}_{\mathcal{Q}} \boxplus \exp(\beta)$$

in $\text{Hom}_{\mathcal{V}\mathcal{E}\mathcal{X}t}(\mathcal{Q} \boxplus \mathcal{D}, (\mathcal{Q} \boxplus \mathcal{D}) + d\beta)$; more generally,

$$(3.22) \quad \exp(\beta') \boxplus \exp(\beta'') = \exp(\beta' + \beta'')$$

3.3.2. Subtraction. Let \mathcal{A} is a vertex extension of \mathcal{L} and \mathcal{D} a cdo. In [Bre] it is described how to define a Courant algebroid $\mathcal{A} \boxminus \mathcal{D}$. Let us recall this construction.

Let $\overline{\mathcal{Q}} := \mathcal{A} \times_{\mathcal{T}} \mathcal{D}$ so that a section of $\overline{\mathcal{Q}}$ is a pair (v, x) , $v \in \mathcal{A}$, $x \in \mathcal{D}$ with $\pi(v) = \pi(x)$.

Define operations $\cdot : \mathcal{O}_X \times \overline{\mathcal{Q}} \rightarrow \overline{\mathcal{Q}}$, $[,] : \overline{\mathcal{Q}} \times \overline{\mathcal{Q}} \rightarrow \overline{\mathcal{Q}}$, $\langle , \rangle : \overline{\mathcal{Q}} \times \overline{\mathcal{Q}} \rightarrow \mathcal{O}_X$, $\pi : \overline{\mathcal{Q}} \rightarrow \mathcal{T}$, and $\partial : \mathcal{O}_X \rightarrow \overline{\mathcal{Q}}$ as follows:

$$(3.23) \quad a \cdot (v, x) := (a_{(-1)}v, a_{(-1)}x)$$

$$(3.24) \quad [(v, x), (v', x')] := (v_{(0)}v', x_{(0)}x')$$

$$(3.25) \quad \langle (v, x), (v', x') \rangle := v_{(1)}v' - x_{(1)}x'$$

$$(3.26) \quad \pi((v, x)) := \pi(v) = \pi(x)$$

$$(3.27) \quad \partial a = (\partial a, 0)$$

Define $\mathcal{A} \boxminus \mathcal{D}$ to be the pushout of $\overline{\mathcal{Q}}$ with respect to the subtraction map $- : \Omega^1 \times \Omega^1 \rightarrow \Omega^1$.

One can show that all operations defined above make sense on $\mathcal{A} \boxminus \mathcal{D}$. One has

Theorem 3.11. ([Bre], Lemma 5.6) *The sheaf $\mathcal{A} \boxminus \mathcal{D}$ with the operations defined above is a Courant algebroid*

3.3.3. Compatibility of \boxplus and \boxminus .

Theorem 3.12. *The functors*

$$- \boxminus \mathcal{D} : \mathcal{CExt}_{\mathcal{L}}^{(\cdot)}(U) \rightarrow \mathcal{V}\mathcal{E}\mathcal{X}t_{\mathcal{L}}^{(\cdot)}(U)$$

and

$$- \boxplus \mathcal{D} : \mathcal{CExt}_{\mathcal{L}}^{(\cdot)}(U) \rightarrow \mathcal{V}\mathcal{E}\mathcal{X}t_{\mathcal{L}}^{(\cdot)}(U)$$

are mutually inverse equivalences of $\mathcal{G}\mathcal{R}(\Omega^{[2,3]}(U))$ -torsors.

Proof. The compatibility of $\boxplus \mathcal{D}$ and $\boxminus \mathcal{D}$ with $\mathcal{G}\mathcal{R}(\Omega^{[2,3]}(U))$ -action follows from properties (3.21 - 3.22) and their obvious analogs for \boxminus . Let us construct the natural isomorphisms $\eta_{\mathcal{A}} : \mathcal{A} \simeq (\mathcal{A} \boxminus \mathcal{D}) \boxplus \mathcal{D}$ where \mathcal{A} is a vertex extension of \mathcal{L} and \mathcal{D} is a cdo.

Define

$$\eta_{\mathcal{A}}(v) = ((v, x), x)$$

where $x \in \mathcal{D}$ is arbitrary.

To show $\eta_{\mathcal{A}}$ is well-defined note that for any $x, y \in \mathcal{D}$ with $\pi(x) = \pi(y) = \pi(v)$ we have $x - y \in \Omega^1$ and

$$((v, x), x) = ((v, (y-x)+y), x) = ((v, y)+(y-x), x) = ((v, y), x+(y-x)) = ((v, y), y)$$

To verify $\eta_{\mathcal{A}}$ is a morphism we check

$$\begin{aligned} a_{(-1)}((v, x), x) &= (a(v, x), a_{(-1)}x) = ((a_{(-1)}v, a_{(-1)}x), a_{(-1)}x) = \eta_{\mathcal{A}}(a_{(-1)}v) \\ ((v, x), x)_{(0)}((v', x'), x') &= ([(v, x), (v', x')], x_{(0)}x') = ((v_{(0)}v', x_{(0)}x'), x_{(0)}x') = \eta_{\mathcal{A}}(v_{(0)}v') \\ ((v, x), x)_{(1)}((v', x'), x') &= \langle (v, x), (v', x') \rangle + x_{(1)}x' = v_{(1)}v' - x_{(1)}x' + x_{(1)}x' = v_{(1)}v' \end{aligned}$$

To check that $\eta_{\mathcal{A}}$ is an isomorphism, one can check that the map $\Psi : (\mathcal{A} \boxminus \mathcal{D}) \boxplus \mathcal{D} \rightarrow \mathcal{A}$, $((v, x), y) \mapsto v + (y - x)$. is a well-defined inverse to Ψ . (Note that every section $((v, x), y)$ of $(\mathcal{A} \boxminus \mathcal{D}) \boxplus \mathcal{D}$ can be written as $((v, x), y) = ((v + (y - x), x + (y - x)), y) = ((v + (y - x), y), y)$ with $v + (y - x)$ independent of the choice of representative $((v, x), y)$).

The construction of the natural isomorphisms $\eta'_{\mathcal{Q}} : \mathcal{Q} \rightarrow \mathcal{Q} \boxplus \mathcal{D} \boxminus \mathcal{D}$ is analogous and left to the reader. \square

The constructions of sections 3.3.1, 3.3.2 and Theorem 3.12 furnish the proof of Theorem 3.9.

3.4. Classification.

3.4.1. *Local existence.* Let U be suitable for chiralization and suppose $\nabla : \mathcal{T}_U \rightarrow \mathcal{L}_U$ is a connection.

Theorem 3.13. *Then the following are equivalent:*

- (1) *The category $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot, \cdot)}(U)$ is nonempty*
- (2) *The category $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot, \cdot)}(U)$ is nonempty*
- (3) *The Pontryagin form $\frac{1}{2}\langle c(\nabla) \wedge c(\nabla) \rangle$ is exact.*

Proof. Since U is suitable for chiralization, there exists a CDO \mathcal{D} on X . Then (1) and (2) are equivalent due to the addition / subtraction operations: given a vertex extension \mathcal{A} there exists a Courant extension $\mathcal{Q} = \mathcal{A} \boxminus \mathcal{D}$ and vice versa, given \mathcal{Q} one can produce a vertex extension $\mathcal{A} = \mathcal{Q} \boxplus \mathcal{D}$. Finally, the equivalence of (2) and (3) is the content of Theorem 3.6. \square

3.4.2. The obstruction.

Theorem 3.14. *Suppose $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot, \cdot)}$ is nonempty. Then its class is equal to*

$$cl(\mathcal{VE}xt_{\mathcal{L}}^{(\cdot, \cdot)}) = -\frac{1}{2}p_1(\mathcal{L}, \langle \cdot, \cdot \rangle) + ch_2(\Omega_X^1)$$

where $p_1(\mathcal{L}, \langle \cdot, \cdot \rangle)$ is the first Pontryagin class of a Lie algebroid \mathcal{L} with pairing $\langle \cdot, \cdot \rangle$.

Proof. What we will be proving is the following:

$$cl(\mathcal{VE}xt_{\mathcal{L}}^{(\cdot, \cdot)}) = cl(\mathcal{CE}xt_{\mathcal{L}}^{(\cdot, \cdot)}) + cl(\mathcal{CD}\mathcal{O}(X))$$

This is indeed sufficient, in view of Theorem 3.8 and the fact that $cl(\mathcal{CD}\mathcal{O}(X)) = ch_2(\Omega_X^1)$ [GMS1, Bre]

Let \mathfrak{U} be a cover of X by open subsets U suitable for chiralization.

Since $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$ is nonempty, so is $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}$. Suppose we are given a trivialization of the gerbe \mathcal{CDO} and that of $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}$.

In other words, we are given a CDO \mathcal{D}_i and a Courant extension \mathcal{Q}_i on each U_i , as well as isomorphisms

$$\eta_{ij} : \mathcal{D}_i|_{U_{ij}} \longrightarrow \mathcal{D}_j|_{U_{ij}} + \alpha_{ij}^{ch}$$

and

$$\theta_{ij} : \mathcal{Q}_i|_{U_{ij}} \longrightarrow \mathcal{Q}_j|_{U_{ij}} + \alpha_{ij}^Q$$

where $\alpha_{ij}^Q, \alpha_{ij}^{ch} \in \Omega^{3,cl}(U_{ij})$, such that on triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ one has

$$\alpha_{ij}^{ch} + \alpha_{jk}^{ch} = \alpha_{ik}^{ch}, \quad \alpha_{ij}^Q + \alpha_{jk}^Q = \alpha_{ik}^Q,$$

and

$$(3.28) \quad \eta_{jk}\eta_{ij}\eta_{ik}^{-1} = \exp(\beta_{ijk}^{ch}), \quad \theta_{jk}\theta_{ij}\theta_{ik}^{-1} = \exp(\beta_{ijk}^Q),$$

for some $\beta_{ijk}^{ch}, \beta_{ijk}^Q \in \Omega^2(U_{ijk})$

Then $(\alpha_{ij}^{ch}, \beta_{ijk}^{ch})$ and $(\alpha_{ij}^Q, \beta_{ijk}^Q)$ are cocycles representing the classes of the gerbes \mathcal{CDO}_X and $\mathcal{CE}xt_{\mathcal{L}}^{(\cdot)}$ respectively.

Now let us construct a trivialization of the gerbe $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$. Define

$$\mathcal{A}_i = \mathcal{Q}_i \boxplus \mathcal{D}_i \in \mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}(U_i).$$

One has the following isomorphisms:

$$\mathcal{Q}_i|_{U_{ij}} \boxplus \mathcal{D}_i|_{U_{ij}} \xrightarrow{\theta_{ij} \boxplus \eta_{ij}} (\mathcal{Q}_j + \alpha_{ij}^Q)|_{U_{ij}} \boxplus (\mathcal{D}_j + \alpha_{ij}^{ch})|_{U_{ij}} = \mathcal{Q}_j|_{U_{ij}} \boxplus \mathcal{D}_j|_{U_{ij}} + (\alpha_{ij}^Q + \alpha_{ij}^{ch})$$

the latter being the identity on the level of vector spaces, by definition of $\boxplus \alpha$ (cf. sect. 3.1.1).

Thus

$$\theta_{ij} \boxplus \eta_{ij} : \mathcal{A}_i \xrightarrow{\sim} \mathcal{A}_j + (\alpha_{ij}^Q + \alpha_{ij}^{ch}).$$

The collection $(\mathcal{A}_i, (\alpha_{ij}^Q + \alpha_{ij}^{ch}), \theta_{ij} \boxplus \eta_{ij})$ is a trivialization of the gerbe $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$.

Let us compute its class.

On triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ we have (cf. (3.28), (3.20), (3.22))

$$(3.29) \quad \begin{aligned} (\theta_{jk} \boxplus \eta_{jk})(\theta_{ij} \boxplus \eta_{ij})(\theta_{ik} \boxplus \eta_{ik})^{-1} &= \theta_{jk}\theta_{ij}\theta_{ik}^{-1} \boxplus \eta_{jk}\eta_{ij}\eta_{ik}^{-1} \\ &= \exp(\beta_{ijk}^Q) \boxplus \exp(\beta_{ijk}^{ch}) \\ &= \exp(\beta_{ijk}^Q + \beta_{ijk}^{ch}) \end{aligned}$$

(here, again, we slightly abuse the notation by writing θ_{ij} for any of its translates under the action of $Gr(\Omega^{[2,3]})$).

It follows that $(\alpha_{ij}^Q + \alpha_{ij}^{ch}, \beta_{ijk}^Q + \beta_{ijk}^{ch})$ is a cocycle representing the class of the gerbe $\mathcal{VE}xt_{\mathcal{L}}^{(\cdot)}$. \square

4. DEFORMATION OF TWISTED CDO

4.1. Twisted chiral differential operators. In this section we recall the definition of the sheaf $\mathcal{D}_X^{ch,tw}$ of twisted chiral differential operators (TCDO) corresponding to a given CDO \mathcal{D}^{ch} on a smooth projective variety X .

4.1.1. The universal Lie algebroid \mathcal{T}^{tw} . The Lie algebroid underlying TCDO is a “family of all TDO”. More precisely, the universal enveloping algebra \mathcal{D}_X^{tw} of \mathcal{T}^{tw} possesses the following property: for every $\lambda \in H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$ there exists an ideal $\mathfrak{m}_\lambda \subset \mathcal{D}_X^{tw}$ such that the quotient $\mathcal{D}_X^{tw}/\mathfrak{m}_\lambda$ is isomorphic to the tdo \mathcal{D}_X^λ corresponding to the class λ .

Let us sketch the construction.

Since X is projective, $H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$ is finite-dimensional, and there exists an affine cover \mathfrak{U} so that $\check{H}^1(\mathfrak{U}, \Omega_X^1 \rightarrow \Omega_X^{2,cl}) = H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$.

Let $\Lambda = \check{H}^1(\mathfrak{U}, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$. We fix a lifting $\check{H}^1(\mathfrak{U}, \Omega_X^1 \rightarrow \Omega_X^{2,cl}) \rightarrow \check{Z}^1(\mathfrak{U}, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$ and identify the former with the subspace of the latter defined by this lifting. Thus, each $\lambda \in \Lambda$ is a pair of cochains $\lambda = ((\lambda_{ij}^{(1)}), (\lambda_i^{(2)}))$ with $\lambda_{ij}^{(1)} \in \Omega^1(U_i \cap U_j)$, $\lambda_i^{(2)} \in \Omega^{2,cl}(U_i)$, satisfying $d_{DR}\lambda_{ij}^{(1)} = d_{\check{C}}\lambda_i^{(2)}$ and $d_{\check{C}}\lambda_{ij}^{(1)} = 0$.

For $\lambda = (\lambda_{ij}^{(1)}, \lambda_i^{(2)}) \in \Lambda$ denote \mathcal{D}^λ the corresponding sheaf of twisted differential operators. One can consider \mathcal{D}^λ as an enveloping algebra of the (Picard) Lie algebroid $\mathcal{T}^\lambda = \mathcal{D}_1^\lambda$ [BB2]. As an \mathcal{O}_X -module, \mathcal{T}^λ is an extension

$$0 \rightarrow \mathcal{O}_X \mathbf{1} \rightarrow \mathcal{T}^\lambda \rightarrow \mathcal{T}_X \rightarrow 0$$

given by $(\lambda_{ij}^{(1)})$. The Lie algebra structure on $\mathcal{T}_{U_i}^\lambda$ is given by $[\xi, \eta]_{\mathcal{T}^\lambda} = [\xi, \eta] + i_\xi i_\eta \lambda_i^2 \mathbf{1}$. and $[\mathbf{1}, \mathcal{T}_{U_i}^\lambda] = 0$.

Let $\{\lambda_i^*\}$ and $\{\lambda_i\}$ be dual bases of Λ^* and Λ respectively. Denote by k the dimension of Λ .

Define \mathcal{T}^{tw} to be an abelian extension

$$0 \rightarrow \mathcal{O}_X \otimes \Lambda^* \rightarrow \mathcal{T}_X^{tw} \rightarrow \mathcal{T}_X \rightarrow 0$$

such that $[\Lambda^*, \mathcal{T}^{tw}] = 0$ and there exist connections $\nabla_i : \mathcal{T}_{U_i} \rightarrow \mathcal{T}_{U_i}^{tw}$ satisfying

$$(4.1) \quad \nabla_j(\xi) - \nabla_i(\xi) = \sum_r \iota_\xi \lambda_r^{(1)}(U_{ij}) \lambda_r^*$$

$$(4.2) \quad [\nabla_i(\xi), \nabla_i(\eta)] - \nabla_i([\xi, \eta]) = \sum_r \iota_\xi \iota_\eta \lambda_r^{(2)}(U_i) \lambda_r^*$$

It is clear that the pair $(\mathcal{T}^{tw}, \mathcal{O}_X \otimes \Lambda^* \hookrightarrow \mathcal{T}^{tw})$ is independent of the choices made.

We call the universal enveloping algebra $\mathcal{D}_X^{tw} = U_{\mathcal{O}_X}(\mathcal{T}^{tw})$ the universal sheaf of twisted differential operators.

4.1.2. A universal twisted CDO. Let $ch_2(X) = 0$ and fix a CDO \mathcal{D}_X^{ch} . To each such sheaf one can attach a universal twisted CDO, $\mathcal{D}_X^{ch,tw}$, a sheaf of vertex algebras whose “underlying” Lie algebroid is \mathcal{T}_X^{tw} . Let us place ourselves in the situation of the previous section, where we had a fixed affine cover $\mathfrak{U} = \{U_i\}$ of a projective algebraic manifold X , dual bases $\{\lambda_i\} \in H^1(X, \Omega_X^{[1,2>})$, $\{\lambda_i^*\} \in H^1(X, \Omega_X^{[1,2>})^*$, and a lifting $H^1(X, \Omega_X^{[1,2>}) \rightarrow Z^1(\mathfrak{U}, \Omega_X^{[1,2>})$.

We can assume that U_i are suitable for chiralization. Let us fix, for each i , an abelian basis $\tau_1^{(i)}, \tau_2^{(i)}, \dots$ of $\Gamma(U_i, \mathcal{T}_X)$. Then the CDO \mathcal{D}^{ch} is given by a collection of 3-forms $\alpha^{(i)} \in \Gamma(U_i, \Omega_X^{3,cl})$ (cf. sect. 2.5, Theorem 2.1) and transition maps $g_{ij} : \mathcal{D}_{U_j}^{ch}|_{U_i \cap U_j} \rightarrow \mathcal{D}_{U_i}^{ch}|_{U_i \cap U_j}$. Let us as well fix splittings $\mathcal{T}_{U_i} \hookrightarrow \mathcal{D}_{U_i}^{ch}$ and view g_{ij} as maps $g_{ij} : (\mathcal{T}_{U_j} \oplus \Omega_{U_j}^1)|_{U_i \cap U_j} \rightarrow (\mathcal{T}_{U_i} \oplus \Omega_{U_i}^1)|_{U_i \cap U_j}$

The universal sheaf of twisted chiral differential operators $\mathcal{D}_X^{ch,tw}$ corresponding to \mathcal{D}_X^{ch} is a vertex envelope of the \mathcal{O}_X -vertex algebroid \mathcal{A}^{tw} determined by the following:

- \mathcal{A}^{tw} is a vertex extensions of $(\mathcal{T}_X^{tw}, 0)$;
- there are embeddings $\mathcal{T}_{U_i} \hookrightarrow \mathcal{A}_{U_i}$ such that

$$\tau_l^{(i)}(0)\tau_m^{(i)} = \iota_{\tau_l^{(i)}}\iota_{\tau_m^{(i)}}\alpha^{(i)} + \sum \iota_{\tau_l^{(i)}}\iota_{\tau_m^{(i)}}\lambda_k^{(2)}(U_i)\lambda_k^*$$

- the transition function from U_j to U_i is given by

$$(4.3) \quad g_{ij}^{tw}(\xi) = g_{ij}(\xi) - \sum \iota_\xi \lambda_k^{(1)}(U_i \cap U_j)\lambda_k^*$$

See [AChM] for a detailed construction.

4.1.3. *Locally trivial twisted CDO.* Observe that there is an embedding

$$(4.4) \quad H^1(X, \Omega_X^{1,cl}) \hookrightarrow H^1(X, \Omega_X^{[1,2>})$$

The space $H^1(X, \Omega_X^{1,cl})$ classifies *locally trivial* twisted differential operators, those that are locally isomorphic to \mathcal{D}_X . Thus for each $\lambda \in H^1(X, \Omega_X^{1,cl})$, there is a unique up to isomorphism TDO $\overset{\circ}{\mathcal{D}}_X^\lambda$ such that for each sufficiently small open $U \subset X$, $\overset{\circ}{\mathcal{D}}_X^\lambda|_U$ is isomorphic to \mathcal{D}_U . Let us see what this means at the level of the universal TDO.

In terms of Čech cocycles the image of embedding (4.4) is described by those $(\lambda^{(1)}, \lambda^{(2)})$, see section 4.1.1, where $\lambda^{(2)} = 0$, and this forces $\lambda^{(1)}$ to be closed. Picking a collection of such cocycles that represent a basis of $H^1(X, \Omega_X^{1,cl})$ we can repeat the constructions of sections 4.1.1 and 4.1.2 to obtain sheaves $\overset{\circ}{\mathcal{T}}_X^{tw}$ and $\overset{\circ}{\mathcal{D}}_X^{ch,tw}$. The latter is glued of pieces isomorphic (as vertex algebras) to $\mathcal{D}_{U_i}^{ch} \otimes H_X$ with transition functions as in (4.3); here H_X is the vertex algebra of differential polynomials on $H^1(X, \Omega_X^{1,cl})$. We will call the sheaf $\overset{\circ}{\mathcal{D}}_X^{ch,tw}$ the *universal locally trivial sheaf of twisted chiral differential operators*.

4.2. **TCDO on flag manifolds.** Let us see what our constructions give us if $X = \mathbb{P}^1$. We have $\mathbb{P}^1 = \mathbb{C}_0 \cup \mathbb{C}_\infty$, a cover $\mathfrak{U} = \{\mathbb{C}_0, \mathbb{C}_\infty\}$, where \mathbb{C}_0 is \mathbb{C} with coordinate x , \mathbb{C}_∞ is \mathbb{C} with coordinate y , with the transition function $x \mapsto 1/y$ over $\mathbb{C}^* = \mathbb{C}_0 \cap \mathbb{C}_\infty$.

Defined over \mathbb{C}_0 and \mathbb{C}_∞ are the standard CDOs, $\mathcal{D}_{\mathbb{C}_0}^{ch}$ and $\mathcal{D}_{\mathbb{C}_\infty}^{ch}$. The spaces of global sections of these sheaves are polynomials in $\partial^n(x)$, $\partial^n(\partial_x)$ (or $\partial^n(y)$, $\partial^n(\partial_y)$ in the latter case), where ∂ is the translation operator, so that, cf. sect. 2.5,

$$(\partial_x)_{(0)}x = (\partial_y)_{(0)}y = 1.$$

There is a unique up to isomorphism CDO on \mathbb{P}^1 , $\mathcal{D}_{\mathbb{P}^1}^{ch}$; it is defined by gluing $\mathcal{D}_{\mathbb{C}_0}^{ch}$ and $\mathcal{D}_{\mathbb{C}_\infty}^{ch}$ over \mathbb{C}^* as follows [MSV]:

$$(4.5) \quad x \mapsto 1/y, \quad \partial_x \mapsto (-\partial_y)_{(-1)}(y^2) - 2\partial(x).$$

The canonical Lie algebra morphism

$$(4.6) \quad sl_2 \rightarrow \Gamma(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}),$$

where

$$(4.7) \quad e \mapsto \partial_x, \quad h \mapsto -2x\partial_x, \quad f \mapsto -x^2\partial_x,$$

e, h, f being the standard generators of sl_2 , can be lifted to a vertex algebra morphism

$$(4.8) \quad V_{-2}(sl_2) \rightarrow \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{ch}),$$

where

$$(4.9) \quad \begin{aligned} e_{(-1)}|0\rangle &\mapsto \partial_x, \\ h_{(-1)}|0\rangle &\mapsto -2(\partial_x)_{(-1)}x, \\ f_{(-1)}|0\rangle &\mapsto -(\partial_x)_{(-1)}x^2 - 2\partial(x). \end{aligned}$$

The twisted version of all of this is as follows ([AChM]).

Since $\dim \mathbb{P}^1 = 1$,

$$H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1 \rightarrow \Omega_{\mathbb{P}^1}^{2,cl}) = H^1(\Omega_{\mathbb{P}^1}^{1,cl}),$$

so all twisted CDO on \mathbb{P}^1 are locally trivial. Furthermore, $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^{1,cl}) = \mathbb{C}$ and is spanned by the cocycle $\mathbb{C}_0 \cap \mathbb{C}_\infty \mapsto dx/x$. We have $H_{\mathbb{P}^1} = \mathbb{C}[\lambda^*, \partial(\lambda^*), \dots]$. Let $\mathcal{D}_{\mathbb{C}_0}^{ch,tw} = \mathcal{D}_{\mathbb{C}_0}^{ch} \otimes H_{\mathbb{P}^1}$, $\mathcal{D}_{\mathbb{C}_\infty}^{ch,tw} = \mathcal{D}_{\mathbb{C}_\infty}^{ch} \otimes H_{\mathbb{P}^1}$ and define $\mathcal{D}_{\mathbb{P}^1}^{ch,tw}$ by gluing $\mathcal{D}_{\mathbb{C}_0}^{ch,tw}$ onto $\mathcal{D}_{\mathbb{C}_\infty}^{ch,tw}$ via

$$(4.10) \quad \lambda^* \mapsto \lambda^*, \quad x \mapsto 1/y, \quad \partial_x \mapsto -(\partial_y)_{(-1)}y^2 - 2\partial(y) + y_{(-1)}\lambda^*.$$

Morphism (4.8) “deforms” to

$$(4.11) \quad V_{-2}(sl_2) \rightarrow \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{ch,tw}),$$

$$(4.12)$$

$$e_{(-1)}|0\rangle \mapsto \partial_x, \quad h_{(-1)}|0\rangle \mapsto -2(\partial_x)_{(-1)}x + \lambda^*, \quad f_{(-1)}|0\rangle \mapsto -(\partial_x)_{(-1)}x^2 - 2\partial(x) + x_{(-1)}\lambda^*.$$

Furthermore, consider $T = e_{(-1)}f_{(-1)} + f_{(-1)}e_{(-1)} + 1/2h_{(-1)}h \in V_{-2}(sl_2)$. It is known that $T \in \mathfrak{z}(V_{-2}(sl_2))$, the center of $V_{-2}(sl_2)$, and in fact, the center $\mathfrak{z}(V_{-2}(sl_2))$ equals the commutative vertex algebra of differential polynomials in T .

The formulas above show

$$(4.13) \quad T \mapsto \frac{1}{2}\lambda_{(-1)}^*\lambda^* - \partial(\lambda^*) \in H_{\mathbb{P}^1}.$$

All of the above is easily verified by direct computations, cf. [MSV]. The higher rank analogue is less explicit but valid nevertheless.

Let G be a simple complex Lie group, $B \subset G$ a Borel subgroup, $X = G/B$, the flag manifold, $\mathfrak{g} = \text{Lie } G$ the corresponding Lie algebra, \mathfrak{h} a Cartan subalgebra. One has a sequence of maps

$$(4.14) \quad \mathfrak{h}^* \rightarrow H^1(X, \Omega_X^{1,cl}) \rightarrow H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl}).$$

The leftmost map attaches to an integral weight $\lambda \in P \subset \mathfrak{h}^*$ the Chern class of the G -equivariant line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, and then extends thus defined map $P \rightarrow H^1(X, \Omega_X^{1,cl})$ to \mathfrak{h}^* by linearity. The rightmost one is engendered by the standard spectral sequence converging to hypercohomology. It is easy to verify that both these maps are isomorphisms. Therefore,

$$(4.15) \quad \mathfrak{h}^* \xrightarrow{\sim} H^1(X, \Omega_X^{1,cl}) \xrightarrow{\sim} H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl}),$$

and each twisted CDO on X is locally trivial.

Constructed in [MSV] – or rather in [FF1], see also [F1] and [GMS2] for an alternative approach – is a vertex algebra morphism

$$(4.16) \quad V_{-h^\vee}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^{ch}).$$

Furthermore, it is an important result of Feigin and Frenkel [FF2], see also an excellent presentation in [F1], that $V_{-h^\vee}(\mathfrak{g})$ possesses a non-trivial center, $\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))$, which, as a vertex algebra, isomorphic to the algebra of differential polynomials in $r_{\mathfrak{g}}$ variables.

Lemma 4.1. [AChM] *Morphism (4.16) “deforms” to*

$$\rho : V_{-h}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^{ch, tw}).$$

Moreover, $\rho(\mathfrak{z}(V_{-h^\vee}(\mathfrak{g}))) \subset H_X$.

4.3. A deformation.

4.3.1. *Motivation: Wakimoto modules.* Let $X = G/B_-$ be a flag variety and $U = NB_- \subset X$ the big cell of X .

In virtue of Lemma 4.1, the sections $\Gamma(U, \mathcal{D}_X^{ch, tw})$ become a $V_{-h^\vee}(\mathfrak{g})$ -module, hence a \mathfrak{g} -module at the critical level. Following [FF1, F2], we call $\Gamma(U, \mathcal{D}_X^{ch, tw})$ a *Wakimoto module of highest weight* $(0, -h^\vee)$, to be denoted $W_{0, -h^\vee}$.

By construction $W_{0, -h^\vee} = \mathcal{D}^{ch}(U) \otimes H_X$. In fact, Feigin and Frenkel proved [FF1] that there exists a whole family of \mathfrak{g} -modules

$$W_{0, k-h^\vee} = \mathcal{D}^{ch}(U) \otimes H_k$$

where H_k is the Heisenberg vertex algebra associated to the space \mathfrak{h} with bilinear pairing $k\langle , \rangle_0$, i.e., k times the canonically normalized Killing form. The \mathfrak{g} -module structure is defined by a vertex algebra morphism

$$V_{k-h^\vee}(\mathfrak{g}) \rightarrow \mathcal{D}^{ch}(U) \otimes H_k$$

and thus, $W_{0, k-h^\vee}$ is a \mathfrak{g} -module of level $k - h^\vee$.

When the level is critical, $W_{0, -h^\vee} = \Gamma(U_e, \mathcal{D}_X^{ch, tw})$. One might ask whether sheaves with an analogous property exist for Wakimoto modules at a non-critical level. To be more precise, we are interested in a sheaf \mathcal{V} of vertex algebras such that:

- its sections on the big cell U and its W -translates are isomorphic to the tensor product of vertex algebras $\mathcal{D}^{ch}(\mathbb{A}^{\dim \mathfrak{g}/\mathfrak{b}}) \otimes H_k$, for nonzero k ;
- the associated Lie algebroid of \mathcal{V} is the universal tdo \mathcal{T}^{tw} .

In other words, \mathcal{V} is a vertex extension of the pair $(\mathcal{T}_{G/B}^{tw}, k\langle , \rangle_0)$

We show that such sheaves do indeed exist on G/B ; moreover, the construction is rather general and can be carried out for any variety. We call the obtained sheaves the *deformations of TCDO* or *deformed TCDO*; deformations because they depend on \langle , \rangle as a parameter, with $\langle , \rangle = 0$ corresponding to a TCDO.

4.3.2. *Definition.* The discussion above suggests the following definition.

Let X be a smooth projective variety and \mathcal{T}_X^{tw} the Lie algebroid underlying the universal TDO (cf. section 4.1.1). Recall that \mathcal{T}_X^{tw} fits into an exact sequence

$$0 \rightarrow \mathcal{O}_X \otimes \Lambda^* \rightarrow \mathcal{T}_X^{tw} \rightarrow \mathcal{T}_X \rightarrow 0$$

where $\Lambda = H^1(X, \Omega^1 \rightarrow \Omega^{2,cl})$.

Let us fix a symmetric bilinear pairing $\langle , \rangle : \Lambda^* \times \Lambda^* \rightarrow \mathbb{C}$ and extend \mathcal{O}_X -linearly to $\mathcal{O}_X \Lambda^*$.

Definition 4.2. We will say that a sheaf \mathcal{V} is a \langle , \rangle -deformation of TCDO if \mathcal{V} is a vertex extension of the pair $(\mathcal{T}_X^{tw}, \langle , \rangle)$.

Without specifying \langle , \rangle , a deformation of TCDO is just a vertex extension of the Lie algebroid \mathcal{T}_X^{tw} .

Being vertex extensions, \langle , \rangle -deformations form a stack, to be denoted

$$\mathcal{TCDO}_X^{\langle , \rangle} := \mathcal{VExt}_{\mathcal{T}^{tw}}^{\langle , \rangle}$$

4.4. **Classification of deformations.** We apply the results of sections 3.4.2.

Theorem 3.14 implies that, when $\mathcal{TCDO}_X^{\langle , \rangle}$ is locally nonempty, its class is equal to

$$cl(\mathcal{TCDO}_X^{\langle , \rangle}) = cl(\mathcal{CExt}_{\mathcal{T}^{tw}}^{\langle , \rangle}) + ch_2(\Omega_X^1)$$

We are going to use the description of $cl(\mathcal{CExt}_{\mathcal{T}^{tw}}^{\langle , \rangle})$ given in section 3.2.

Let us work in the setup of sections 4.1.1, 4.1.2. Thus, we pick a basis $\{\lambda_r\}$ of $H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$, a dual basis $\{\lambda_r^*\}$ in $H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$, and a lifting $H^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl}) \rightarrow \check{Z}^1(X, \Omega_X^1 \rightarrow \Omega_X^{2,cl})$, so that each λ_r is a pair of cochains $(\lambda_r^{(1)}, \lambda_r^{(2)}) \in \prod \Omega^1(U_{ij}) \times \prod \Omega^{2,cl}(U_i)$.

By construction, the Lie algebroid \mathcal{T}_X^{tw} admits connections $\nabla_i : \mathcal{T}_{U_i} \rightarrow \mathcal{T}_{U_i}^{tw}$ such that

$$(4.17) \quad A_{ij} := \nabla_i - \nabla_j = -\lambda_k^* \lambda_k^{(1)}(U_{ij})$$

(summation over repeated indices is assumed) and

$$(4.18) \quad c(\nabla_i) = -\lambda_k^* \lambda_k^{(2)}(U_i)$$

Theorem 4.3. Let $\langle , \rangle \neq 0$ be a symmetric bilinear form on $\mathcal{O}_X \otimes \Lambda^*$. Then:

(1) \langle , \rangle -deformations exist locally on X if and only if the 4-form

$$(4.19) \quad \langle \lambda_r^*, \lambda_s^* \rangle \lambda_r^{(2)}(U_i) \wedge \lambda_s^{(2)}(U_i)$$

is exact;

(2) Assume (1) and pick, for every i , a 3-form H_i such that $2dH_i = \langle \lambda_r^*, \lambda_s^* \rangle \lambda_r^{(2)}(U_i) \wedge \lambda_s^{(2)}(U_i)$. Denote

$$\alpha_{ij} = \frac{1}{2} \langle \lambda_r^*, \lambda_s^* \rangle \left(\lambda_r^{(2)}(U_i) + \lambda_r^{(2)}(U_j) \right) \wedge \lambda_s^{(1)}(U_{ij}) + H_i - H_j$$

and

$$\beta_{ijk} = \langle \lambda_r^*, \lambda_s^* \rangle \lambda_r^{(1)}(U_{ij}) \wedge \lambda_s^{(1)}(U_{jk})$$

Then a global \langle , \rangle -deformation exists if and only if the class of the cocycle $(\alpha_{ij}, \beta_{ijk})$ in $H^2(X, \Omega_X^2 \rightarrow \Omega_X^{3,cl})$ is equal to $-ch_2(\Omega_X)$ (minus second graded piece of Chern character of Ω_X^1).

Proof. (1) Follows from Theorem 3.13, since the 4-form (4.19) is just the Pontryagin form $\frac{1}{2}\langle c(\nabla_i) \wedge c(\nabla_i) \rangle$ for the Lie algebroid \mathcal{T}_X^{tw} .

(2) Using the connections ∇_i (and formulas (4.17), (4.18)) in the construction of the section 3.2 one verifies that the cocycle $(\alpha_{ij}, \beta_{ijk})$ represents the class of $\mathcal{CE}xt_{\mathcal{T}^{tw}}^{\langle , \rangle}$. The statement follows immediately from Theorem 3.14 and the fact that $cl(\mathcal{CDO}) = ch_2(\Omega_X^1)$ [Bre]. \square

Remark 4.4. In the presence of CDO, the classification problem for deformed TCDO becomes one for Courant extensions of $(\mathcal{T}_X^{tw}, \langle , \rangle)$, as any CDO \mathcal{D}^{ch} defines an equivalence of stacks over X

$$? \boxplus \mathcal{D}^{ch} : \mathcal{CE}xt_{\mathcal{T}^{tw}}^{\langle , \rangle} \rightarrow \mathcal{T}\mathcal{CDO}^{\langle , \rangle}.$$

4.5. Deformations of locally trivial TCDO. Recall from section 4.1.3 that locally trivial TCDO are constructed in the same way as TCDO by consistently replacing $H^1(X, \Omega^1 \rightarrow \Omega^{2,cl})$ with $H^1(X, \Omega^{1,cl})$. In particular we construct a Lie algebroid $\overset{\circ}{\mathcal{T}}^{tw}$.

We define the corresponding versions of deformations as follows. A *locally trivial deformed TCDO* is a vertex extension of $\overset{\circ}{\mathcal{T}}^{tw}$. A *locally trivial \langle , \rangle -deformation of TCDO* is a vertex extension of $(\overset{\circ}{\mathcal{T}}^{tw}, \langle , \rangle)$.

The locally trivial \langle , \rangle -deformations form a stack $\mathcal{T}\mathcal{CDO}^{\langle , \rangle, lt}$.

Theorem 4.3 has the following analogue in the locally trivial case:

Theorem 4.5. *Let $\langle , \rangle \neq 0$ be a symmetric \mathcal{T} -invariant bilinear form on $\mathcal{O}_X \otimes \Lambda^*$. Then:*

(1) *\langle , \rangle -deformations exist locally on X .*

(2) *every \langle , \rangle -deformation $\mathcal{A}_{\langle , \rangle}^{tw, lt}$ is locally isomorphic to $\mathcal{D}_U^{ch} \otimes H_{\langle , \rangle}$ where \mathcal{D}_U^{ch} is a CDO and $H_{\langle , \rangle}$ is a Heisenberg vertex algebra associated to the space $H^1(X, \Omega^{1,cl})^*$ with the bilinear form \langle , \rangle .*

(3) *Denote*

$$\beta_{ijk} = \langle \lambda_r^*, \lambda_s^* \rangle \lambda_r^1(U_{ij}) \wedge \lambda_s^1(U_{jk})$$

and let $[(0, (\beta_{ijk}))]$ stand for the class of $(0, (\beta_{ijk}))$ in $H^2(\Omega^2 \rightarrow \Omega^{3,cl})$.

Then the class of $\mathcal{T}\mathcal{CDO}^{\langle , \rangle, lt}$ in $H^2(\Omega^2 \rightarrow \Omega^{3,cl})$ is given by

$$cl(\mathcal{T}\mathcal{CDO}^{\langle , \rangle, lt}) = ch_2(\Omega_X^1) + [(0, (\beta_{ijk}))]$$

Proof. (1) By construction, the Lie algebroid $\overset{\circ}{\mathcal{T}}^{tw}$ admits flat connections $\nabla_i : \overset{\circ}{\mathcal{T}}_U \rightarrow \overset{\circ}{\mathcal{T}}_X|_{U_i}$, which implies $\langle c(\nabla_i) \wedge c(\nabla_i) \rangle = 0$. The local existence now follows from Theorem 3.13.

(2) Suppose ∇ is a flat connection on an open set $U \subset X$, and let $\mathcal{Q} = \mathcal{Q}_{\nabla, H}$ be a Courant extension of $\overset{\circ}{\mathcal{T}}^{tw}$ over U (cf. 3.2).

Then $\mathcal{Q} \simeq \mathcal{T}_U \oplus (\mathcal{O}_U \otimes H^1(X, \Omega^{1,cl})) \oplus \Omega_U^1$ and since $c(\nabla) = 0$ one immediately observes from (3.9) and (3.10) that the constant subsheaf $H^1(X, \Omega^{1,cl})^*$ “decouples”. It is clear from the construction, that it stays decoupled in $\mathcal{Q} \boxplus \mathcal{D}$, for any cdo \mathcal{D} on U . It has a structure of a Courant (equivalently, vertex) algebroid over $\text{Spec}(\mathbb{C})$ whose vertex envelope is the algebra $H_{\langle , \rangle}$.

(3) The proof is identical to that of Theorem 4.3, Part (2). \square

4.6. Deformed TCDO on \mathbb{P}^1 .

This is a continuation of Example 4.2.
Recall that we are using standard coordinate charts U_0 and U_1 so that $\mathbb{P}^1 = U_0 \cup U_1$ with $0 \in U_0$, $\infty \in U_1$ and coordinate functions $x : U_0 \rightarrow \mathbb{C}$ and $y : U_1 \rightarrow \mathbb{C}$ with $x = \frac{1}{y}$. Denote

$$\lambda = \frac{dy}{y} = -\frac{dx}{x}$$

a cocycle representative of a generator of 1-dimensional $H^1(\mathbb{P}^1, \Omega^{1,cl})$.

By definition,

$$(4.20) \quad \mathcal{T}_{U_i}^{tw} = \mathcal{T}_{U_i} \oplus \mathcal{O}_{U_i} \lambda^*, \quad i = 0, 1,$$

with Lie bracket defined by $[\xi, \eta]_{\mathcal{L}} = [\xi, \eta]$, $[\xi, a\lambda^*] = \xi(a)\lambda^*$.

Let $\nabla_i : \mathcal{T}_{U_i} \rightarrow \mathcal{T}_{U_i}^{tw}$, $i = 0, 1$ be the canonical inclusions. The formula (4.17) in this case reads as

$$(4.21) \quad \nabla_1 - \nabla_0 = \frac{dy}{y} \lambda^*,$$

which dictates the following gluing map $g_{01} : \mathcal{T}_1^{tw}|_{\mathbb{C}^*} \rightarrow \mathcal{T}_0^{tw}|_{\mathbb{C}^*}$

$$(4.22) \quad \begin{aligned} \xi &\mapsto \xi + i_\xi \lambda \cdot \lambda^* \\ \lambda^* &\mapsto \lambda^* \end{aligned}$$

In the chosen coordinates, it is $\partial_y = -x^2 \partial_x + x \lambda^*$.

4.6.1. The deformed TCDO. We wish to construct a vertex extension of $(\mathcal{T}_{\mathbb{P}^1}^{tw}, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is a symmetric \mathcal{T}^{tw} -invariant \mathcal{O} -bilinear pairing on $\mathfrak{g}(\mathcal{T}^{tw}) = \mathcal{O}_X \otimes H^1(X, \Omega^1 \rightarrow \Omega^2)^* = \mathcal{O} \cdot \lambda^*$. In this case it is determined by a number $k \in \mathbb{C}$ assigned to $\langle \lambda^* | \lambda^* \rangle$.

Let us fix k and assume $k \neq 0$ ($k = 0$ corresponds to the usual TCDO).

Since $\dim \mathbb{P}^1 = 1$, $\Omega^i = 0$ for $i > 1$, in particular $H^i(\mathbb{P}^1, \Omega^2 \rightarrow \Omega^{3,cl}) = 0$ for all i . Therefore there exists a unique vertex extension for any pair $(\mathcal{L}, \langle \cdot, \cdot \rangle)$. Let $\mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw}$ denote the vertex extension of $(\mathcal{T}_{\mathbb{P}^1}^{tw}, \langle \cdot, \cdot \rangle)$.

Denote by $H_{\mathbb{P}^1}^{\langle \cdot, \cdot \rangle}$ the Heisenberg vertex algebra generated by a field λ^* satisfying $\lambda^*_{(1)} \lambda^* = \langle \lambda^*, \lambda^* \rangle$, $\lambda^*_{(n)} \lambda^* = 0, n \neq 1$. Theorem 4.5 describes $\mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw}$ locally: one has isomorphisms of vertex algebras $(\mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw})_{U_i} \simeq \mathcal{D}_{U_i}^{ch} \otimes H_{\mathbb{P}^1}^{\langle \cdot, \cdot \rangle}$, $i = 0, 1$. Some global information is provided by the following

Theorem 4.6. (1) *There are isomorphisms $\phi_i : \mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw}|_{U_i} \rightarrow \mathcal{D}_{U_i}^{ch} \otimes H_{\mathbb{P}^1}^{\langle \cdot, \cdot \rangle}$, $i = 0, 1$, such that*

$$(4.23) \quad \phi_0 \phi_1^{-1}(\partial_y) = -x^2 \partial_x - 2dx + x \lambda^* + \frac{1}{2} \langle \lambda^*, \lambda^* \rangle dx$$

$$(4.24) \quad \phi_0 \phi_1^{-1}(\lambda^*) = \lambda^* - \langle \lambda^*, \lambda^* \rangle x^{-1} dx$$

(2) *The anchor map of $\mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw}$ induces a vector space isomorphism*

$$H^0(\mathbb{P}^1, \mathcal{A}_{\langle \cdot, \cdot \rangle}^{tw}) \simeq H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}).$$

Proof. (1) The construction of section 3.3.1 and the results of section 3.2 imply that the most general gluing formula is as follows:

$$(4.25) \quad \xi \mapsto g_{ij}(\xi) + A(\xi) - \frac{1}{2}\langle A(\xi), A \rangle + \iota_\xi \beta$$

$$(4.26) \quad g \mapsto g - \langle g, A \rangle$$

where g_{ij} is a transition function for a CDO, $\beta \in \Omega^2_{U_i \cap U_j}$, $A = \nabla_j - \nabla_i$, for some connectoions $\nabla_i : \mathcal{T}_{U_i} \rightarrow \mathcal{L}_{U_i}$.

Applying to our case and using (4.5) and (4.21), we see that

$$(4.27) \quad \partial_y \mapsto -x^2 \partial_x - 2dx + x\lambda^* + \frac{1}{2}\langle \lambda^*, \lambda^* \rangle dx$$

and the map $\tilde{\mathfrak{g}}|_{U_1} \rightarrow \tilde{\mathfrak{g}}|_{U_0}$ is given by

$$(4.28) \quad \lambda^* \mapsto \lambda^* - \langle \lambda^*, \frac{dy}{y} \lambda^* \rangle = \lambda^* + k \frac{dx}{x}$$

(2) The gluing formula (4.28) implies that the map $H^0(\mathbb{P}^1, \mathfrak{g}) \rightarrow H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1)$ in the long exact sequence associated to $0 \rightarrow \Omega_{\mathbb{P}^1}^1 \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$ is an isomorphism.

Since $H^j(\mathbb{P}^1, \Omega^1) = H^k(\mathbb{P}^1, \mathfrak{g}) = 0$ for $j \neq 1, k \neq 0$, one can conclude that $H^i(\mathbb{P}^1, \tilde{\mathfrak{g}}) = 0$ for all i .

In turn, the long cohomology sequence associated to the sequence

$$0 \rightarrow \tilde{\mathfrak{g}} \rightarrow \mathcal{A}_{\langle , \rangle}^{tw} \rightarrow \mathcal{T}_{\mathbb{P}^1} \rightarrow 0$$

shows that $H^i(\mathbb{P}^1, \mathcal{A}_{\langle , \rangle}^{tw}) \simeq H^i(\mathbb{P}^1, \mathcal{T})$. \square

4.7. Embedding of affine \mathfrak{sl}_2 . For $\kappa \in \mathbb{C}$ let $\mathcal{A}_\kappa(\mathfrak{sl}_2)$ denote the vertex algebroid over \mathbb{C} equal to \mathfrak{sl}_2 as a space, with bracket $g_{(0)}g' = [g, g']$ and pairing $g_{(1)}g' = \kappa \langle g | g' \rangle$ where $\langle \cdot | \cdot \rangle$ is the canonically normalized invariant form (for \mathfrak{sl}_2 , it is $\frac{1}{4}\langle \cdot | \cdot \rangle_{Killing}$).

Let

$$(4.29) \quad \begin{aligned} e &= \partial_x \\ h &= -2\partial_{x(-1)}x + \lambda^* \\ f &= -\partial_{x(-1)}x^2 - 2dx + x\lambda^* + \frac{1}{2}\langle \lambda^* | \lambda^* \rangle dx \end{aligned}$$

Lemma 4.7. *The elements e, f, h given by the formulas (4.29)*

- (1) *satisfy the relations of $\widehat{\mathfrak{sl}}_2(\kappa)$ where $\kappa = \frac{\langle \lambda^* | \lambda^* \rangle}{2} - 2$;*
- (2) *belong to $H^0(\mathbb{P}^1, \mathcal{A}_{\langle , \rangle}^{tw})$*

Proof. Restricted to the big cell, the statement of Part (1) goes back to Wakimoto [W]; see also [F1].

The rest follows from the following equalities over $U_0 \cap U_1$:

$$(4.30) \quad \begin{aligned} \partial_x &= -\partial_{y(-1)}y^2 - 2dy + y\lambda^* + \frac{1}{2}\langle \lambda^* | \lambda^* \rangle dy \\ -2\partial_{x(-1)}x + \lambda^* &= 2\partial_{y(-1)}y - \lambda^* \\ -\partial_{x(-1)}x^2 - 2dx + x\lambda^* + \frac{1}{2}\langle \lambda^* | \lambda^* \rangle dx &= \partial_y \end{aligned}$$

□

Corollary 4.8. *The formulas (4.29) define an isomorphism of vertex algebroids over k*

$$(4.31) \quad \mathcal{A}_\kappa(\mathfrak{sl}_2) \simeq H^0(\mathbb{P}^1, \mathcal{A}_{\langle , \rangle}^{tw})$$

that extends to the vertex algebra embedding

$$(4.32) \quad V_\kappa(\mathfrak{sl}_2) \longrightarrow H^0(\mathbb{P}^1, U(\mathcal{D}_{\langle , \rangle}^{ch,tw}))$$

Proof. The map defined by (4.29) is clearly injective and the first statement follows by dimension count. The restriction of the second map to the big cell was shown in [F1] to be injective. □

4.8. The case of a general flag variety.

Recall that we have an identification

$$(4.33) \quad \bar{\alpha} : \mathfrak{h}^* \simeq H^1(X, \Omega^{1,cl}) \simeq H^2(X, \mathbb{C})$$

In other words, the tdo on G/B are classified by \mathfrak{h}^* . The Lie algebroid $\mathcal{T}_{G/B}^{tw}$ is an extension

$$0 \longrightarrow \mathcal{O}_{G/B} \otimes_{\mathbb{C}} \mathfrak{h} \longrightarrow \mathcal{T}_{G/B}^{tw} \longrightarrow \mathcal{T}_{G/B} \longrightarrow 0$$

A deformation of TCDO is therefore a vertex extension of $(\mathcal{T}_{G/B}^{tw}, \langle , \rangle)$ where \langle , \rangle is a symmetric bilinear pairing $\langle , \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$

We have the following

Theorem 4.9. *Let $X = G/B$. Then the class of $\mathcal{TCDO}^{\langle \rangle}$ is equal to 0 if and only if \langle , \rangle is proportional to the restriction of the Killing form on \mathfrak{h} .*

Proof. First, we find a convenient cocycle representation of the obstruction.

Let $\{\chi_r\}$ be the set of fundamental weights, \mathcal{L}_r the corresponding line bundles over X , \mathcal{D}_{χ_r} algebras of tdo acting on \mathcal{L}_r and T_{χ_r} the corresponding Lie algebroids. Define the cocycles $\mu_r = (\mu_r^{ij}) \in \check{Z}^1(X, \Omega_X^{1,cl})$ corresponding to T_{χ_r} . Then the map (4.33) is the one taking χ_r to the class of (μ_r^{ij}) in $H^1(X, \Omega_X^{1,cl})$.

Take λ_r^* to be the basis of \mathfrak{h} dual to the basis $\{\chi_r\}$.

Using Theorem 4.5 and the existence of CDO on X ([GMS2]), we conclude that the class of $\mathcal{TCDO}^{\langle \rangle}$ is represented by a cocycle $\langle \lambda_r^* | \lambda_s^* \rangle \mu_r^{ij} \wedge \mu_s^{jk}$. Its image under the natural embedding $H^2(X, \Omega^2 \rightarrow \Omega^{3,cl}) \rightarrow H^4(X, \mathbb{C})$ (cf. [GMS2]) equals to that of the element

$$S = \langle \lambda_r^* | \lambda_s^* \rangle \chi_r \cdot \chi_s \in S^2 \mathfrak{h}^*.$$

which naturally corresponds to the form $\langle , \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$.

By [BGG], S becomes zero in $H^4(X, \mathbb{C})$ if and only if S is W -invariant. Therefore, the form \langle , \rangle has to be a multiple of the Killing form. □

4.8.1. Embedding of the affine vertex algebra $\mathcal{V}_k(\mathfrak{g})$.

Let X be a G -variety.

Let $\underline{\mathcal{A}}_k(\mathfrak{g})_X$ denote the constant sheaf with sections equal to \mathfrak{g} , equipped with the structure of a \mathbb{C}_X -vertex algebroid as follows:

$$(4.34) \quad \begin{aligned} x_{(0)}y &= [x, y] \\ x_{(1)}y &= k\langle x | y \rangle \\ \pi &= 0, \quad \partial = 0 \end{aligned}$$

Let \mathcal{A} be a (locally trivial) \langle , \rangle -deformation of TCDO.

Let us assume that there is a Lie algebra morphism

$$(4.35) \quad \alpha : \mathfrak{g} \rightarrow \mathcal{T}_X^{tw}$$

lifting the morphism $\bar{\alpha} : \mathfrak{g} \rightarrow \mathcal{T}_X$ induced by the action of G . (This is the case for $X = G/B$).

Consider the sheaf of homomorphisms of vertex algebroids

$$(4.36) \quad \mathcal{H}om_\alpha(\underline{\mathcal{A}}_k(\mathfrak{g})_X, \mathcal{A})$$

that lift the morphism α .

We are mainly interested in the global sections of this sheaf, as they correspond to embeddings of the vertex algebra $\mathcal{V}_k(\mathfrak{g})$ into the envelope of \mathcal{A} .

Proposition 4.10. *Suppose the image of \mathfrak{g} in \mathcal{T}_X generates \mathcal{T}_X as an \mathcal{O}_X -module. Then the sheaf (4.36), if locally nonempty, is an $\Omega^{2,cl}$ -torsor.*

Proof. Let us work locally on a subset $U \subset X$ small enough to admit an identification $\mathcal{A}|_U \simeq \mathcal{T}_X^{tw}|_U \oplus \Omega_U^1$.

Let $w, w' \in \mathcal{H}om_\alpha(\underline{\mathcal{A}}_k(\mathfrak{g})_X, \mathcal{A})(U)$.

Then $w'(g) = w(g) + \omega(g)$ for some $\omega : \mathfrak{g} \rightarrow \Omega^1$, since the \mathcal{T}^{tw} -component is fixed.

Analysis similar to that in [MSV, GMS1] shows that ω must be given by

$$\omega(g) = \iota_{\alpha(g)}\beta$$

where $\beta \in \Omega_X^{2,cl}$.

Conversely, adding $\iota_{\alpha(-)}\beta$ to any $w \in \mathcal{H}om_\alpha(\underline{\mathcal{A}}_k(\mathfrak{g})_X, \mathcal{A})(U)$ gives an element of $\mathcal{H}om_\alpha(\underline{\mathcal{A}}_k(\mathfrak{g})_X, \mathcal{A})(U)$. The statement follows. \square

Remark 4.11. When $\dim X = 1$ the torsor (4.36) is trivial, therefore the existence of local embeddings implies the existence of a global one. For a general flag variety we do not know whether the torsor (4.36) is trivial, but we believe it is.

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